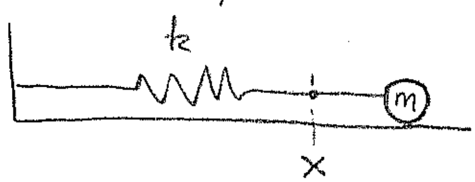
LECTURE 18

- We study the energy stored in a spring w/o friction. The concept of PE-graph analysis is introduced and used to examine the spring motion and... GRAVITY. The idea of studying the potential plane naturally leads to the phase or Poincare plane which plays an important role in qualitative analysis of physical systems. The potential plane idea is important to Quantum Mechanics (QM) and also (as an application of QM) chemistry.

<u>Mathematical Aside</u>: (for breadth, not req^{d})

We examine sol^{n} of the frictionless spring (with stiffness k) and mass (m) in one dimension



$$F_{spring} = -kx$$

$$ma = -kx$$

As a differential eq^{n}: $m\ddot{x} + kx = 0$ where $a = \ddot{x}$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

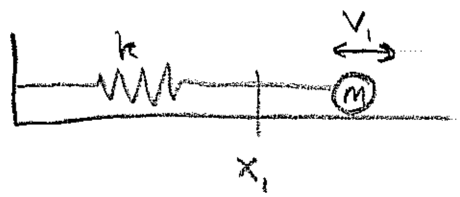
$$\Rightarrow \lambda^2 + \frac{k}{m} = 0 \Longrightarrow \lambda = \pm i\sqrt{\frac{k}{m}}$$

$$\Rightarrow x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$\Rightarrow \boxed{\begin{array}{l} x(t) = A\cos(\omega t + \phi) \\ \text{where } \omega = \sqrt{\frac{k}{m}} = \text{angular frequency} \\ \phi = \text{phase angle} \\ A = \text{Amplitude} \end{array}}$$

while math not req^{d}, you ought to know the facts in box for your own confidence of problem solving

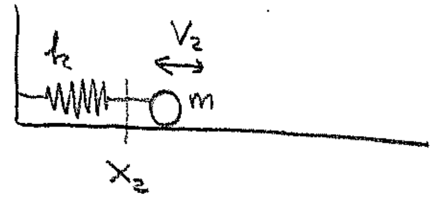
Let's study energy in spring/mass problem



BEGINNING

x_1 and V_1 could be positive or negative or even zero

$$E_1 = \frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2$$



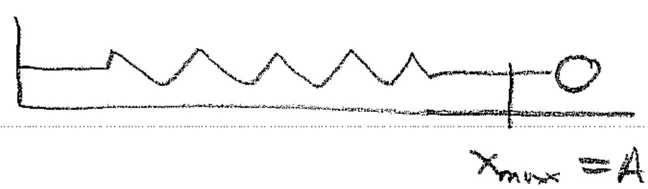
END

x_2 and V_2 could be positive or negative or even zero

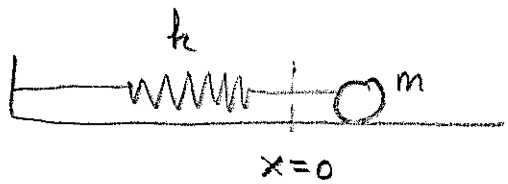
$$E_2 = \frac{1}{2}kx_2^2 + \frac{1}{2}mV_2^2$$

Energy Conservation is valid here because $F_{net} = -kx$ and $F_{net} = -\frac{dU}{dx}$ is conservative. The key idea to solve problems is $E_1 = E_2$ so if I give you any three of x_1, x_2, V_1, V_2 you can use energy conservation to find the fourth.

$\boxed{\text{E1}}$ Suppose that $A = x_{max}$ then what is $|V_{max}|$? (for a spring with stiffness k and mass m

① 

Concept: when at max length we have $v = 0 \Rightarrow KE_1 = 0$.
$\hookrightarrow E_1 = \frac{1}{2}kA^2$

② 

Concept: as we cross equilibrium there is zero PE.

$$E_2 = \frac{1}{2}mV_{max}^2$$

Thus $E_1 = E_2 \Rightarrow \frac{1}{2}kA^2 = \frac{1}{2}mv^2 \Rightarrow \underline{V_{max} = \sqrt{\frac{kA^2}{m}}}$.

(I did an example with #5 in Lecture)

Potential Energy Graphical Analysis:

Suppose $F_{net} = -\frac{dU}{dx}$ for a one-dimensional system with coordinate x and potential energy U. We have Newton's 2^{nd} Law:

$$m\frac{dV}{dt} = -\frac{dU}{dx} \rightsquigarrow m\frac{dv}{dt} + \frac{dU}{dx} = 0$$

Multiply by v to obtain ($v = dx/dt$)

$$mV\frac{dV}{dt} + \frac{dx}{dt}\frac{dU}{dx} = 0$$

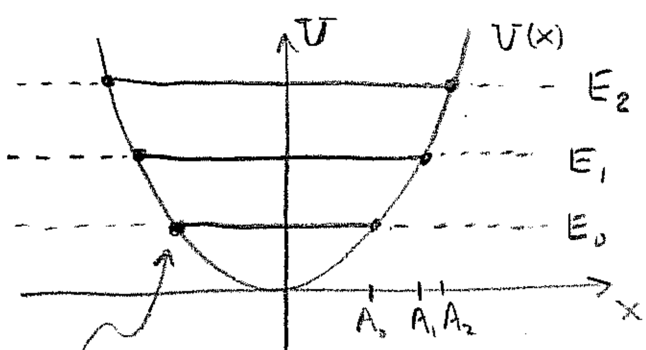
$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}mV^2\right) + \frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mV^2 + U\right) = 0$$

Thus, $\boxed{E = \frac{1}{2}mV^2 + U = \text{constant}}$.

We proved this already by more general arguments but, I like the calculation above so here it stays.

<u>CONCEPT</u>: KE is non-negative: $K = \frac{1}{2}mV^2 \geq 0$. It follows $E(x,V) = K(V) + U(x) \geq U(x)$. This means the graph of $E(x,V) = E_0$ only exists where the PE $U(x) \leq E_0$. When $U(x) < E_0$ there is some amount of KE, when $U(x) = E_0$ then $KE = 0$. (Utilizing this concept is <u>potential plane analysis</u>)

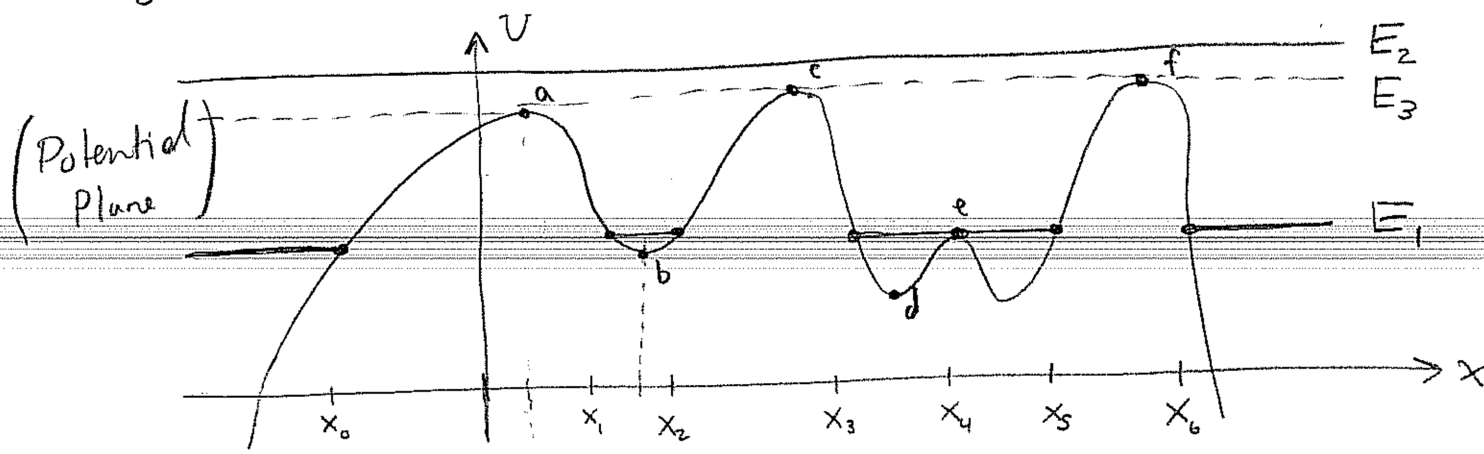
$\boxed{E2}$ The spring has $U(x) = \frac{1}{2}kx^2$



for E_0, $x_{max} = A_0$
for E_1 $x_{max} = A_1$
for E_2 $x_{max} = A_2$

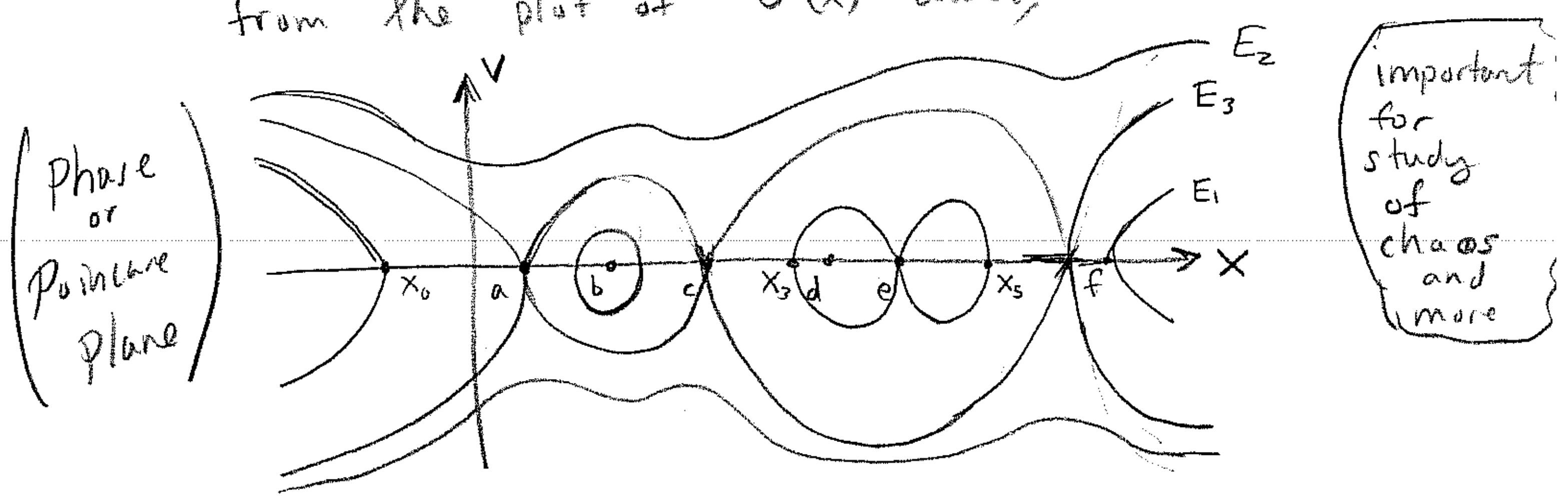
$E_0 = U(-A_0)$ and $KE = 0$ at the interection point.

$\boxed{E3}$ Analyze motion for $U(x)$ given below, assume system is conservative,



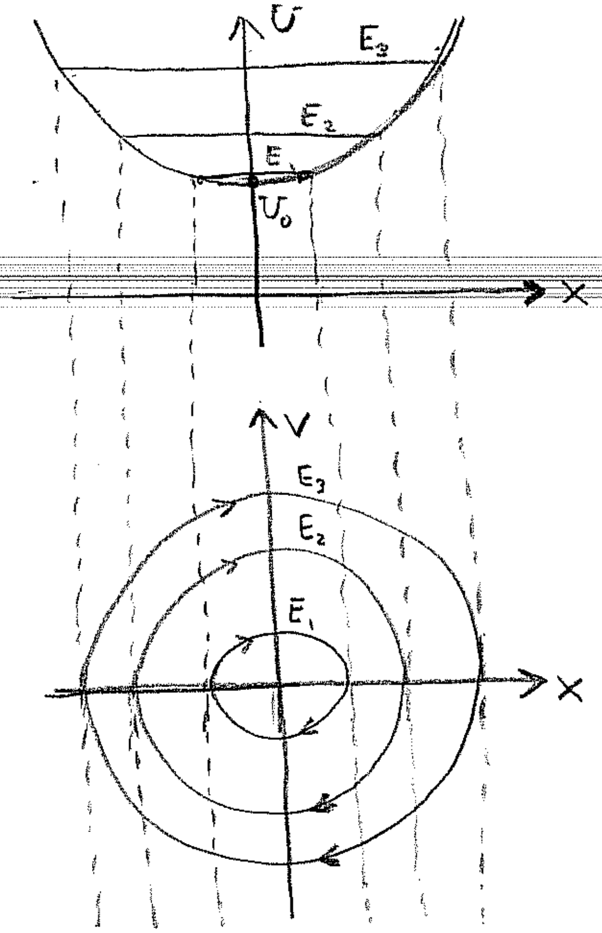
- If particle has $E = E_1$ then motion in $[x_0, x_1]$, $[x_2, x_3]$ and $[x_5, x_6]$ are <u>unphysical</u> since those regions $\Rightarrow KE < 0$ which is impossible!
  - there are unbounded orbits $x \leq x_0$ and $x \geq x_6$
  - there are bounded orbits in-between the hills.
  - the point x_4 is unstable equilibrium.

We might be helped by graphing orbits in the <u>phase plane</u>. The following is created from the plot of $U(x)$ above,



- You can study both x and v at once and these are the <u>level curves</u> of the energy function.

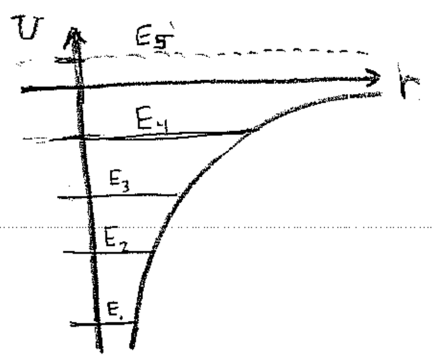
E4 Draw potential plane and phase plane for $U(x) = \frac{1}{2}kx^2 + U_0$



Potential Plane
critical points at x with $dU/dx = 0$.

Phase Plane
critical points will be on x-axis directly under critical points for U.

E5 The potential for gravity is $U(r) = \frac{-GmM}{r}$ where r = distance from planet to sun (or earth to satelite ...)



I'll leave the phase plane to your imagination here, you can see $E_1, E_2, E_3, E_4 < 0$ give bounded orbits whereas $E_5 > 0$ is unbounded.

Remark: I have a lengthy mathematical examination of this material in my DEq$^{n's}$ website. Ask if interested!