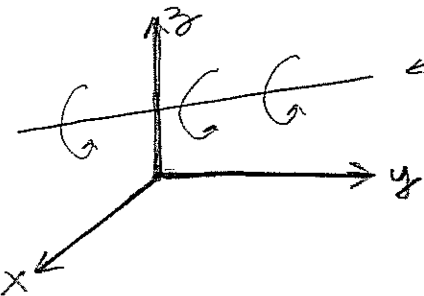
LECTURE 25:

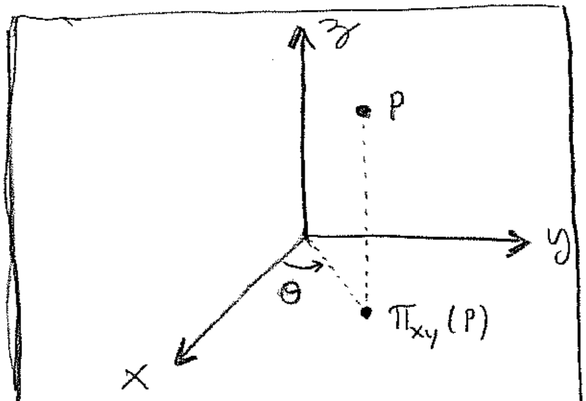
- We discuss the fundamentals of rotational dynamics in this lecture. We follow Chapter 9 of Tipler, note the vector nature of rotations is exposed in Chapter 10. For now, we consider rotations about an axis and this makes our problem 1-dimensional in a sense. We have derived a few facts about circular motion in LECTURE 7.

- An axis of rotation is the one set of points which remains fixed under rotations of arbitrary angle



axis doesn't move under a rotation.

Usually I like to think about rotational motion around the z-axis, we want to think about



$$\underbrace{\Pi_{xy}(P) = (P\cdot\hat{i})\hat{i} + (P\cdot\hat{j})\hat{j}}_{\text{projection onto } xy\text{-plane.}}$$

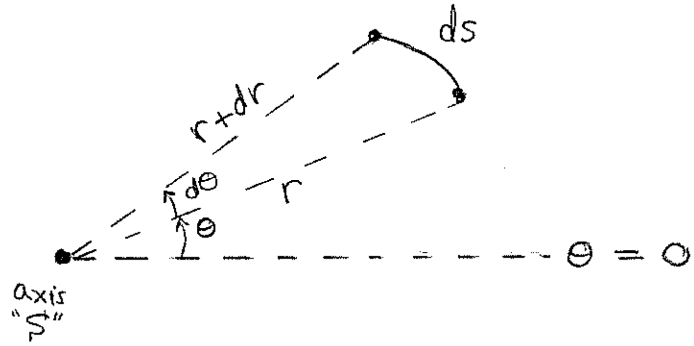
how θ changes for P as time evolves. We take $\theta = 0$ on the x-axis an measure θ CCW by projecting $P = (P_x, P_y, P_z) \longrightarrow (P_x, P_y, 0)$

θ defined as the simultaneous $\text{sol}^{\underline{n}}$ to $\dfrac{P_x}{\sqrt{P_x^2+P_y^2}} = \cos\theta$ and

$$\frac{P_y}{\sqrt{P_x^2+P_y^2}} = \sin\theta.$$

Since we're mainly interested in two-dimensional problems I'll turn our focus there now

Radian Measure and Angular Velocity or Acceleration



$r \approx r + dr$

$ds = r\,d\theta$

$$\boxed{\omega \overset{\text{def}^{n}}{=\!=} \frac{d\theta}{dt} = \text{angular velocity with respect to axis } S}$$

Notice $\omega > 0 \Rightarrow$ CCW motion [Counter Clock Wise ↺]
$\omega < 0 \Rightarrow$ CW motion [Clock Wise ↻]

Remark: the axis with which we measure ω may not be fixed in place. Sometimes they move, like the axel on a car in motion or the rotor on a helicopter. Even if the helicopter was accelerating in all sorts of directions it still is meaningful to ask how many revolutions were made by the blades over some time period. Or, you might think about twirling a Yo-Yo around. We could use our arm (from elbow down) as the axis. Once we pick some initial angle and direction then we can quantify ω for the Yo-Yo provided we avoid one very special case. ($r = 0$ troubling)

$$\boxed{\alpha \overset{\text{def}^{n}}{=\!=} \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \text{angular acceleration with respect to axis } S}$$

Notice that $ds = r\,d\theta \Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt} \therefore \boxed{V_T = r\omega}$

Likewise, $\frac{dV}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} \therefore \boxed{a_T = r\alpha}$

(need to assume r = constant for the eq^{n}'s above)

On Relating linear and rotational velocities

Suppose our particle is moving with angle θ (a function of time) and radial distance r (also a function of time in general). How is $\omega = \frac{d\theta}{dt}$ related to the speed v?

To begin note we have by the very def$^{\underline{n}}$ of r, θ

$$\vec{r}(t) = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right)\hat{i} + \left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)\hat{j}$$

Hence,

$$V = \sqrt{(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2 + (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2}$$

$$= \sqrt{\dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\sin^2\theta\,\dot{\theta}^2 + \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\cos^2\theta\,\dot{\theta}^2}$$

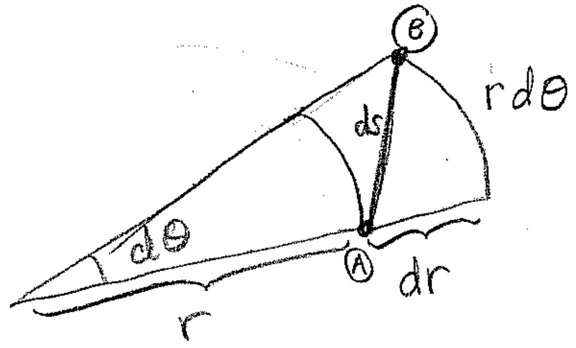
$$= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

That is $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$ or $\underbrace{ds^2 = dr^2 + r^2 d\theta^2}_{\text{not hard to derive geometrically}}$

We find $\boxed{V^2 = V_r^2 + r^2\omega^2}$

If the motion is circular then $r = \text{constant}$ thus $\dot{r} = 0$ or $V_r = 0$. This simplifies things considerably, $\boxed{V = r|\omega|}$ but we relax our demand $v = |\vec{v}|$ here and usually just write $\boxed{V = r\omega}$ with the understanding $V < 0 \Rightarrow$ CW motion.

Remark: see my Coriolis Notes if you'd like to think more about the case $V_r \neq 0$ and what that entails for $\vec{a}$... Also, $\boxed{\text{Lecture 7}}$ derives much of the formulae above.



distance from A to B is ds and infinitesimally

$$ds^2 = dr^2 + r^2 d\theta^2$$

Thus if we divide by dt^2 then we obtain $\frac{ds^2}{dt^2} = \frac{dr^2}{dt^2} + r^2\frac{d\theta^2}{dt^2}$

or $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$ as we derived through more careful argument on the previous page.

Summary:

If θ is the radians rotated about some given axis and r = distance from axis to particle in question then we define (for the particle)

$$\omega = \frac{d\theta}{dt} \quad \text{(angular velocity)}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad \text{(angular acceleration)}$$

If the motion is on a circle of radius r then can state that:

$s = r\theta$ (arclength subtended from $\theta = 0 \longrightarrow \theta = \theta$)

$v = r\omega$ (linear velocity, positive for θ increasing, negative for θ decreasing)

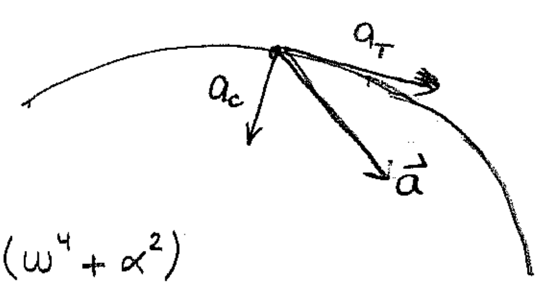
$a_T = r\alpha$ (tangential acceleration)

We can show for circular motion that

$$\vec{a} = a_c\hat{r} + a_T\hat{\theta}$$

$$a_c = -v^2/r = -r\omega^2$$

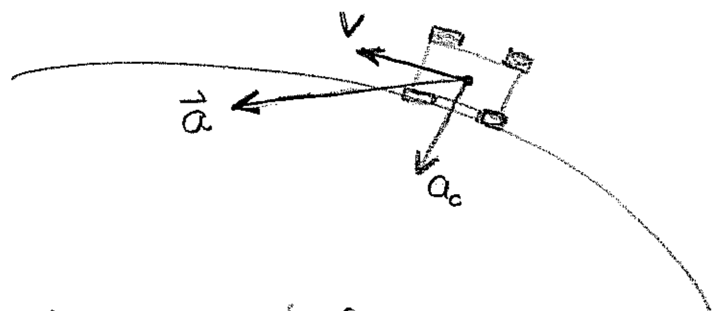
$$a_T = r\alpha$$



Generally, $a^2 = a_c^2 + a_T^2 = r^2(\omega^4 + \alpha^2)$

The fact that $a = r\sqrt{\omega^4 + \alpha^2}$ for circular motion is important if we consider motion along circular arcs at non-constant speed.

E1 Suppose we are accelerating around a turn with radius $r = 30m$. If our speed is increasing at $1\,m/s^2$ then what is the maximum speed we can reach without losing traction on a surface with $\mu_s = 0.8$?



$$m\vec{a} = \underbrace{-mg\hat{k} + \vec{N}}_{\substack{\text{into or} \\ \text{out of} \\ \text{the page} \\ \text{we have}}} + \underbrace{\vec{F}_f}_{\substack{\text{allows} \\ \text{for} \\ \text{motion} \\ \text{in circle.}}}$$

$N = mg$ (flat track) $\qquad F_f = \mu_s mg$

Because the vertical forces cancel we find that $m\vec{a} = \vec{F}_f$ thus

$$\mu_s mg = |m\vec{a}|$$

$$\Rightarrow \mu_s g = a = r\sqrt{\omega^4 + \alpha^2} = \sqrt{r^2\omega^4 + \underbrace{r^2\alpha^2}_{a_T^2}}$$

$$\Rightarrow \mu_s r g = \sqrt{r^4\omega^4 + r^2 a_T^2}$$

$$\Rightarrow v^4 = (\mu_s r g)^2 - r^4\alpha^2 = [(0.8)(30m)(9.81\tfrac{m}{s^2})]^2 - (30m)^2(1\tfrac{m}{s^2})^2$$

$$\Rightarrow v^4 = 54532\,\frac{m^4}{s^4} \quad \therefore \quad \boxed{V = 15.28\,m/s}$$

Details aside, the concept here is that $\vec{F}_f$ is shared between the tasks of maintaining circular motion (a_c) and increasing speed (a_T).

Remark: in real roads and racetracks the banking of turns increases the possible speeds.

Constant Angular Acceleration

Suppose $\alpha = \alpha_o$ where $\frac{d\alpha}{dt} = 0$ then we can integrate twice w.r.t. time and derive,

$$\boxed{\begin{array}{l} \omega(t) = \omega_o + \alpha_o t \quad , \text{ where } \omega(0) = \omega_o \\ \hline \theta(t) = \theta_o + \omega_o t + \frac{1}{2}\alpha_o t^2 \quad , \text{ where } \theta_o = \theta(0) \end{array}}$$

As in the linear motion case we can do a substitution trick to derive a timeless eq$^{\underline{n}}$,

$$\alpha = \frac{d\omega}{dt} = \frac{d\theta}{dt}\frac{d\omega}{d\theta} = \omega\frac{d\omega}{d\theta}$$

$$\Rightarrow \quad \alpha_o d\theta = \omega d\omega$$

separation of variables! will cover in calculus II if not already then soon..

$$\Rightarrow \quad \alpha_o(\theta_f - \theta_o) = \frac{1}{2}(\omega_f^2 - \omega_o^2)$$

$$\therefore \quad \boxed{\omega_f^2 = \omega_o^2 - 2\alpha_o(\theta_f - \theta_o)}$$

E2 A car accelerates from rest to 40 m/s using wheels with 40cm diameter. If the linear acceleration was constant and took 5 seconds then find the angular velocity and acceleration of the wheels. Also how many revolutions do the wheels make?

The angular velocity ω is related to linear velocity v by $v = \omega r$ and here $r = 40\text{cm}/2 = 0.2\text{m}$. It follows $\omega_o = 0$ and $\omega_f = 40\,\text{m/s} / 0.2\text{m} = 200\ \text{rad/s}$. We can calculate

$$\alpha_o = \frac{\omega_f - \omega_o}{t} = \frac{200\ \text{rad/s}}{5\,s} = \boxed{40\ \text{rad/s}^2 = \alpha_o}$$

It follows that $\boxed{\omega(t) = \left(40\ \frac{\text{rad}}{s^2}\right)t}$ and

$$\theta(t) = \theta_o + \left(20\ \frac{\text{rad}}{s^2}\right)t^2 \Rightarrow \theta - \theta_o = \left(20\ \frac{\text{rad}}{s^2}\right)(5s)^2 = 500\ \text{rad}.$$

Now 500 radians $= (500\ \text{rad})\left(\frac{\text{revolution}}{2\pi\ \text{rad}}\right) = \boxed{79.58\ \text{revs}}$

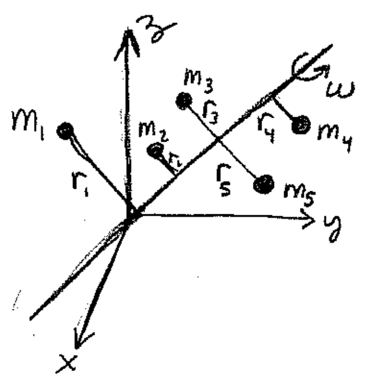
Rotational Kinetic Energy for finitely many Particles:

Imagine a collection of particles $m_1, m_2, .., m_N$ with velocities $V_1, V_2, .., V_N$ as a result of rotating with angular velocity ω at radii $r_1, r_2, ..., r_N$. We assume the masses are somehow fixed in place about a central point which is motionless, this means all of the KE stems from the rotation,

$$KE_{TOTAL} = \sum_{j=1}^{N} \frac{1}{2} m_j V_j^2 \quad : \quad V_j = r_j \omega \quad \forall j = 1, 2, .., N.$$

$$= \sum_{j=1}^{N} \frac{1}{2} m_j r_j^2 \omega^2$$

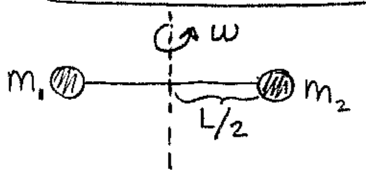
$$= \frac{1}{2} \underbrace{\left[\sum_{j=1}^{N} m_j r_j^2 \right]}_{I} \omega^2 \qquad \therefore \boxed{KE_{rotational} = \frac{1}{2} I \omega^2}$$



I the moment of intertia for $\{m_1, m_2, .., m_N\}$ so described. here $r_1, r_2, .., r_N$ are the radial distances to the masses rotating.

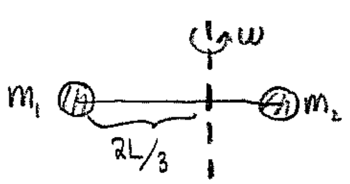
E3 Find KE for barbell rotated around center and then at the 1/3 way down from one mass m, length is L

A)



$$I_A = m_1 \left(\frac{L}{2}\right)^2 + m_2 \left(\frac{L}{2}\right)^2 = \frac{1}{2} m L^2 \Rightarrow KE_A = \frac{mL^2\omega^2}{4}$$

B)



$$I_B = m_1 \left(\frac{2L}{3}\right)^2 + m_2 \left(\frac{L}{3}\right)^2 = \frac{5}{9} m L^2 \Rightarrow KE_B = \frac{5mL^2\omega^2}{18}$$

Question: around which point should we rotate at ω to give the barbell maximum KE?