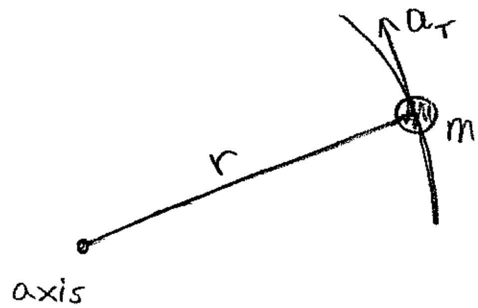
Lecture 27:

- In Lectures 25 & 26 we set the stage. Those Lectures describe rotational kinematics. We now turn to the question of dynamics. We seek to describe the cause (torque) and effect (rotational motion). As in the case of kinematics we'll see the laws of rotational motion follow from the corresponding laws of linear motion. Consider a single particle of mass m,



$$ma_T = F_T \leftarrow \text{tangential component of net-force.}$$

$$\Rightarrow m r \alpha = F_T$$

We expect $I = mr^2$ to play role like m so multiply by r to obtain

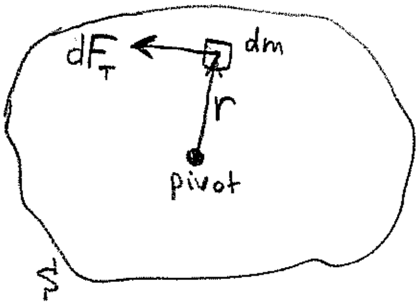
$$mr^2\alpha = r F_T$$

Define $r F_T = \tau$ torque.

and we find $\boxed{\tau = I\alpha}$

(rotational analogue of $F = ma$)

If we consider a distribution of mass rotating about some pivot point then we can generalize this result in a natural manner



$$(r^2 dm)\alpha = r\, dF_T = \text{torque on } dm$$

$$\underbrace{\int_S r\, dF_T}_{} = \int_S r^2 dm\, \alpha = \left(\int_S r^2 dm\right)\alpha$$

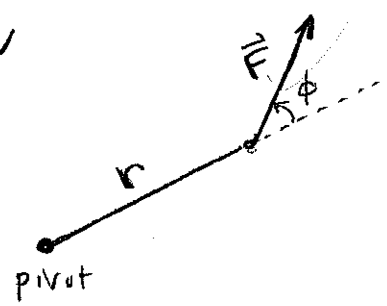
$$\boxed{\text{net torque} = I\alpha}$$

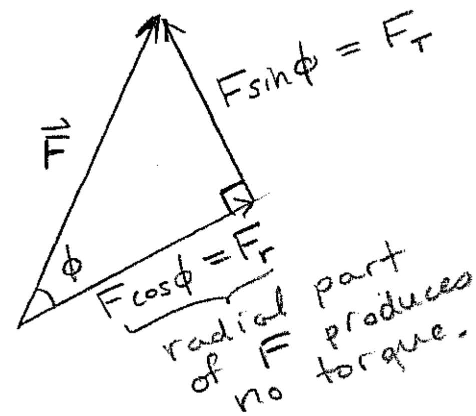
For system of particles $m_1, m_2, \ldots, m_N$ at $r_1, r_2, \ldots, r_N$ have $\boxed{\sum_{j=1}^{N} \tau_j = \left(\sum_{j=1}^{N} m_j r_j^2\right)\alpha}$

The best way to think about torque is given in Chapter 10, but I'll behave and follow Tipler. The key question here is what is F_T for a given problem.

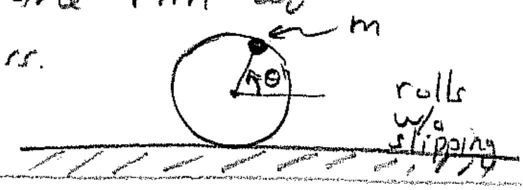
E1 Apply a force $\vec{F}$ at an angle ϕ off the radial line from the pivot point. If $\vec{F}$ is applied at r from pivot what is F_T ? τ_F ?

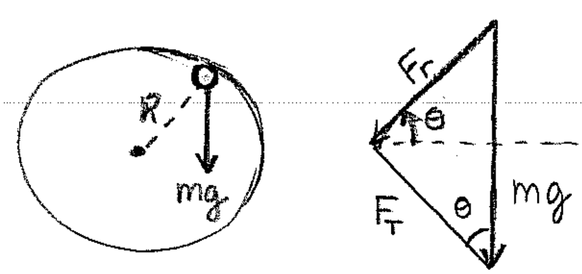
(taking CCW as positive direction)





E2 Find torque on cylinder of mass M and radius R if a small mass m is placed on the rim at an angle θ pictured below. Discuss.





$F_r = mg\sin\theta$

$F_T = mg\cos\theta$

$F_T = 0$ for $\theta = 90°$

$F_T = mg$ for $\theta = 0°$

$F_T = -mg$ for $\theta = 180°$

Thus $\boxed{\tau = -mgR\cos\theta}$ the dynamical eq^{n} is

$$\tau = I\alpha = \Big(\underbrace{\tfrac{1}{2}MR^2}_{\text{cylinder}} + \underbrace{mR^2}_{\text{little mass on rim at } R.}\Big)\alpha$$

$$\frac{1}{2}(M + 2m)R^2\frac{d^2\theta}{dt^2} = -mgR\cos\theta$$

nonlinear ODE, I can only solve small range of θ

E2 Continued

We found $\frac{1}{2}(M+2m)R^2\frac{d^2\theta}{dt^2} = -mgR\cos\theta$.

I can solve this over small regions by approximating $\cos\theta$ with its Taylor series. For example,

$$\theta \approx 0 \implies \cos\theta \approx 1 - \frac{1}{2}\theta^2$$

So to first order in θ we have $\cos\theta \approx 1$ hence

$$\frac{1}{2}(M+2m)R^2\frac{d^2\theta}{dt^2} = -mgR$$

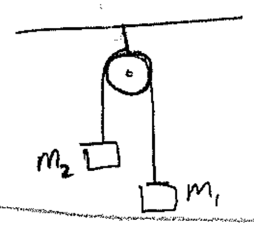
(can solve if want, but it's more interesting to note)

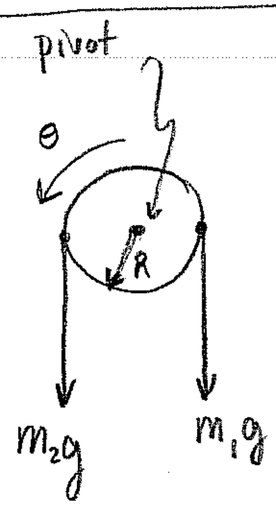
$$\underbrace{R\frac{d^2\theta}{dt^2}} = \frac{-mg}{\frac{1}{2}(M+2m)}$$

$$R\alpha = \boxed{a_T = \frac{-2mg}{M+2m}}$$

Notice as $m \longrightarrow 0$ we find $a_T \longrightarrow 0$
and as $M \longrightarrow 0$ we find $a_T \longrightarrow -g$
Do these limiting cases surprise you? Why doesn't gravity place a nonzero torque on the cylinder w/o the extra mass m?

E3 Find acceleration of an Atwood machine with a massive pulley over which the rope pulls w/o slipping.
Let the $I_{pulley} = \frac{1}{2}MR^2$





Let the pulley be the system and note $\tau_{ext} = \tau_1 + \tau_2$.

$$\tau_1 = -m_1 g R$$

$$\tau_2 = m_2 g R$$

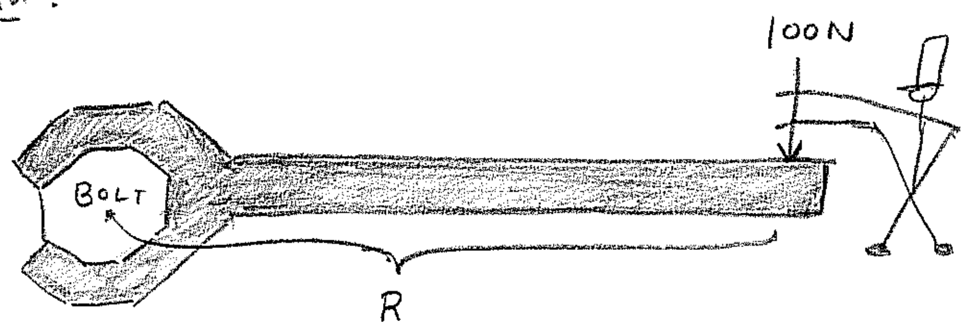
$$\frac{1}{2}MR^2\alpha = m_2 g R - m_1 g R$$

$$\boxed{a_T = R\alpha = \frac{2(m_2 - m_1)g}{M}}$$

E4 Suppose you apply 100N to a wrench. What is the maximum torque you can place on rusty bolt?

The question you should ask at the beginning of any question is "does this make sence ?" My intuition is no. Why? Because the longer the wrench the bigger the torque. Thus w/o additional info the best positive answer would devolve into an argument about what the largest physically plausible wrench is.

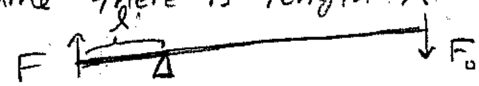
Math :



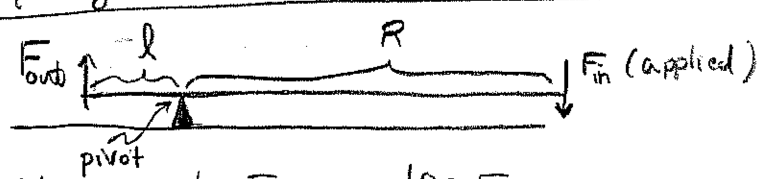
$$\tau = R\,(100N)$$

As $R \longrightarrow \infty$ we see $\tau \longrightarrow \infty$.

E5 How long a lever do you need to increase your applied force by 100 times? Assume there is length l on the left side of the pivot point



(neglect mass of lever for ease of argument here)



If $\tau = 0$ on the pivot point then the $R F_{in} = l F_{out}$.

We want $F_{out} = 100 F_{in}$

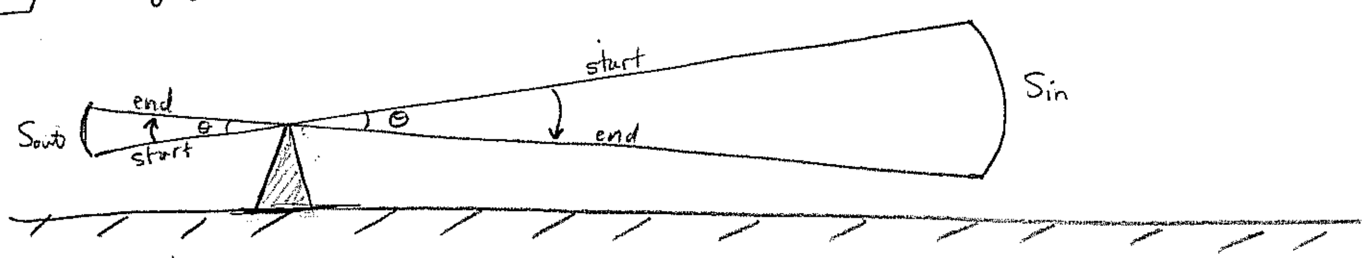
Hence, $l \cdot 100 F_{in} = R \cdot F_{in} \Rightarrow \boxed{R = 100\, l}$

You need a lever 101 times as long as you place the pivot point from the applied F_{out}.

Paradox: we can increase force many times over with a lever (this is called mechanical advantage). Work is (force)(distance) thus we are creating energy from nothing. We can seemingly multiply our work by just adding a lever. Similar machines can be constructed with systems of pulleys. An input of 30 N can multiply to thousands of Newtons.

RESOLUTION: the input force is applied over a proportionally greater distance. The same energy goes in as goes out (assuming no energy is lost to friction...)

$\boxed{E6}$ analyze the work done in $\boxed{E5}$ by F_{in} vs. F_{out}



Note that $W_{out} = -S_{out} F_{out} = -S_{out}\, 100 F_{in}$ whereas $W_{in} = S_{in} F_{in}$ to relate W_{out} & W_{in} we need to link the distances S_{out} and S_{in}. This is accomplished by $S_{out} = l\theta$ and $S_{in} = 100 l \theta$ as the angle is the same in both wedges.
Thus $S_{in} = 100\, S_{out}$ and it follows $W_{out} = -W_{in}$ hence $W_{net} = 0$ which is logical since we insisted $\tau_{net} = 0 \Rightarrow \alpha = 0 \Rightarrow a_T = 0 \Rightarrow \Delta KE = 0$ $\Rightarrow W_{net} = 0$.

Remark: $\tau = r F_T$ hence $[\tau] = (\text{distance})(\text{force}) = \text{energy}$ torque has units of work.

TRANSLATING LINEAR $\leftrightarrow$ ROTATIONAL PHYSICS

linear	rotational	nonslip connections
x position	θ angle	$x = R\theta$
$V = \frac{dx}{dt}$ velocity	$\omega = \frac{d\theta}{dt}$ angular velocity	$V = R\omega$
$a = \frac{dV}{dt} = \frac{d^2x}{dt^2}$ acceleration	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ angular acceleration	$a_T = R\alpha$
m mass	I inertia	
F force	τ torque	
$P = \vec{F}\cdot\vec{v}$ power from $\vec{F}$	$P = \tau\omega$ power from τ	
$KE_{TRANS.} = \frac{1}{2}mv^2$	$KE_{ROT.} = \frac{1}{2}I\omega^2$	
for $a = a_o$ (constant)	for $\alpha = \alpha_o$ (constant)	
$V_f = V_o + a_o t$	$\omega_f = \omega_o + \alpha_o t$	
$x_f = x_o + \omega_o t + \frac{1}{2}a_o t^2$	$\theta_f = \theta_o + \omega_o t + \frac{1}{2}\alpha_o t^2$	
$V_f^2 = V_o^2 + 2a_o(x_f - x_o)$	$\omega_f^2 = \omega_o^2 + 2\alpha_o(\theta_f - \theta_o)$	
$V_{avg} = \frac{1}{2}(V_o + V_f)$	$\omega_{avg} = \frac{1}{2}(\omega_o + \omega_f)$	

In the lecture that follows we simply explore further examples of Chapter 9 material. You should expect the table above to grow to include the rotational analogue of momentum $\vec{P}$, not surprisingly we'll call it <u>angular momentum</u> $\vec{L}$ and as $\vec{F} = \frac{d\vec{P}}{dt}$ it's also the case $\vec{\tau} = \frac{d\vec{L}}{dt}$ ...

To complete this Lecture I once more return to $\boxed{E2}$ this time I explore sol$^{\underline{n}}$ near $\theta = -90°$

$\boxed{E7}$ In $\boxed{E2}$ we found $\frac{1}{2}(M+2m)R\frac{d^2\theta}{dt^2} = -mg\cos\theta$ describes the motion of a cylinder of mass M with a small mass m attached to its outer edge.

$\theta = -90^\circ = -\frac{\pi}{2}$ radians.

$$f(\theta) = \cos\theta, \quad f'(\theta) = -\sin\theta$$

$$f(\theta) = \underbrace{f(-\pi/2)}_{0} + \underbrace{f'(\pi/2)}_{1}(\theta + \tfrac{\pi}{2}) + \cdots = \theta + \frac{\pi}{2} + \cdots$$

I'm applying Taylor's Th^{m} which you will all learn by the semester's end in calculus II.

Continuing,

$$\frac{d^2\theta}{dt^2} = \frac{-2mg}{R(M+2m)}\cos\theta \cong -w^2(\theta + \pi/2)$$

Let $y = \theta + \pi/2$ then $\frac{d^2y}{dt^2} = \frac{d^2\theta}{dt^2}$ and we have

$$\frac{d^2y}{dt^2} = -w^2 y$$

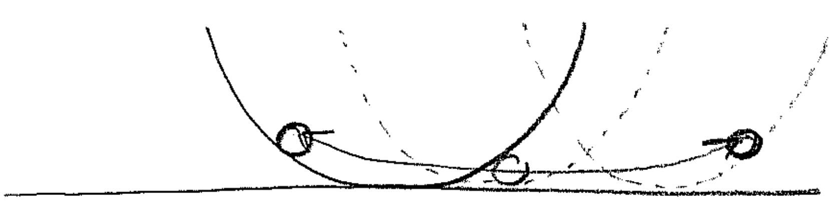
This is easily solved by $y(t) = c_1\cos wt + c_2\sin wt$

Hence, $\theta = -\frac{\pi}{2} + y$ yields

$$\boxed{\theta(t) = -\frac{\pi}{2} + c_1\cos(wt + \phi)}$$

where $w = \sqrt{\frac{2mg}{R(M+2m)}}$ the period of these small oscillations is given by $wT = 2\pi$

$$T = 2\pi\sqrt{\frac{R(M+2m)}{2mg}}$$



cylinder rolls back and forth like a pendulum.