Lecture 29:

- In this lecture we introduce the cross-product. In this course we need the cross-product to understand the geometry of rotations and torque. Generally the cross-product has many uses, a few we discuss here for context. It is important to understand the cross-product both computationally and geometrically. We introduce several "right-hand-rules". These help us describe certain orientations both this semester and the next.

Dot Products: (for contrasting & comparing with cross-product)

Let $\vec{V}, \vec{W}$ be vectors then $\vec{V} \cdot \vec{W} = VW\cos\theta$

or, in components, $\vec{V} \cdot \vec{W} = V_1W_1 + V_2W_2 + V_3W_3$.

The dot-product takes in two vectors and returns a scalar which measures the angle between the vectors ($\theta = \cos^{-1}\left[\dfrac{\vec{V}\cdot\vec{W}}{VW}\right]$ is useful for calculating 3D-angles)

Another important use for dot-products is projection, note,

$$\vec{V} \cdot \hat{i} = V_1 \quad \text{(component of } \vec{V} \text{ in 1-direction)}$$
$$\vec{V} \cdot \hat{j} = V_2 \quad \text{(comp. in 2-direction)}$$
$$\vec{V} \cdot \hat{k} = V_3 \quad \text{(comp. in 3-direction)}$$

Generalizing a bit,

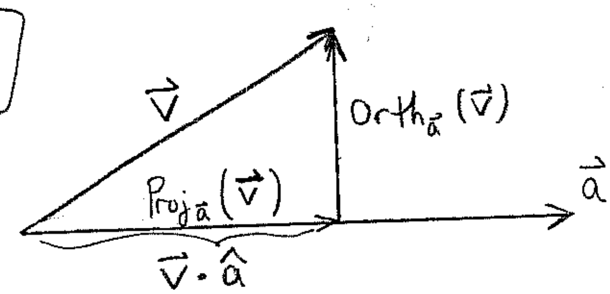
$$\boxed{\text{Comp}_{\vec{a}}(\vec{V}) = \vec{V} \cdot \hat{a} = V_a = \text{length of } \vec{V} \text{ in the } \vec{a}\text{-direction}}$$

The vector component of $\vec{V}$ in the $\hat{a}$ direction is

part of $\vec{V}$ along $\vec{a}$ $\longrightarrow$ $\boxed{\text{Proj}_{\vec{a}}(\vec{V}) = (\vec{V} \cdot \hat{a})\hat{a}}$

$\longrightarrow$ $\boxed{\text{Orth}_{\vec{a}}(\vec{V}) = \vec{V} - (\vec{V} \cdot \hat{a})\hat{a}}$

part of $\vec{V}$ $\perp$ to $\vec{a}$



Continuing: the projection and orthogonal complements are useful devices to break a given vector into pieces $\parallel$ & $\perp$ to another given vector, $\vec{V} = \mathrm{Proj}_{\vec{a}}(\vec{V}) + \mathrm{Orth}_{\vec{a}}(\vec{V})$ and the geometry of this is illustrated on last page.

CROSS PRODUCT:

Let $\vec{V}, \vec{W}$ be vectors then $\vec{V} \times \vec{W}$ is another vector which is $\perp$ to both $\vec{V}$ and $\vec{W}$. In particular, we define for
$$\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}$$
$$\vec{W} = W_x\hat{\imath} + W_y\hat{\jmath} + W_z\hat{k},$$

$$\boxed{\vec{V}\times\vec{W} = (V_yW_z - V_zW_y)\hat{\imath} + (V_zW_x - V_xW_z)\hat{\jmath} + (V_xW_y - V_yW_x)\hat{k}}$$

We observe that we have $\vec{V}\times\vec{W} = -\vec{W}\times\vec{V}$ and a short calculation reveals that,

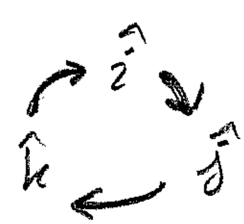
$$(\vec{V}+\vec{W})\times\vec{u} = \vec{V}\times\vec{u} + \vec{W}\times\vec{u}$$
$$(c\vec{V})\times\vec{W} = \vec{V}\times(c\vec{W}) = c\,(\vec{V}\times\vec{W}) \quad \text{for } c\in\mathbb{R}.$$

Moreover, a less short calculation reveals

$$\boxed{\vec{V}\times\vec{W} = VW\sin\theta\,\hat{n} \quad \text{where } \hat{n} \text{ is given by the Right-Hand-Rule (RHR)}}$$

Quintessential examples,

$$\hat{\imath}\times\hat{\jmath} = \hat{k} \qquad \hat{\jmath}\times\hat{\imath} = -\hat{k}$$
$$\hat{\jmath}\times\hat{k} = \hat{\imath} \qquad \hat{k}\times\hat{\jmath} = -\hat{\imath}$$
$$\hat{k}\times\hat{\imath} = \hat{\jmath} \qquad \hat{\imath}\times\hat{k} = -\hat{\jmath}$$



E1 $(\hat{\imath}+3\hat{\jmath})\times(\hat{k}-\hat{\imath}) = \hat{\imath}\times\hat{k} - \cancel{\hat{\imath}\times\hat{\imath}}^{\,0} + 3\hat{\jmath}\times\hat{k} - 3\hat{\jmath}\times\hat{\imath}$

$$= \boxed{-\hat{\jmath} + 3\hat{\imath} + 3\hat{k}}$$

$\boxed{E2}$ $\vec{v} = \langle 1, 3, 0 \rangle = \hat{i} + 3\hat{j}$

$\vec{w} = \langle -1, 0, 1 \rangle = -\hat{i} + \hat{k}$

Another popular notation for $\vec{v} \times \vec{w}$ is the determinant formula below,

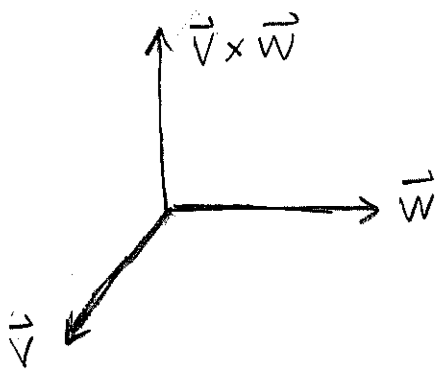
$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

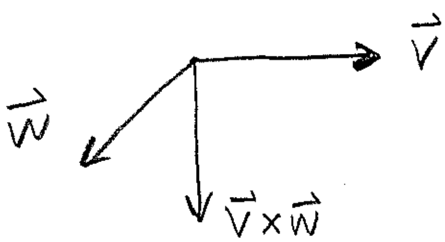
$$= \hat{i}\,(3-0) - \hat{j}\,(1-0) + \hat{k}\,(0+3)$$

$$= \boxed{3\hat{i} - \hat{j} + 3\hat{k}}$$

(same answer as in $\boxed{E1}$)

RIGHT HAND RULE





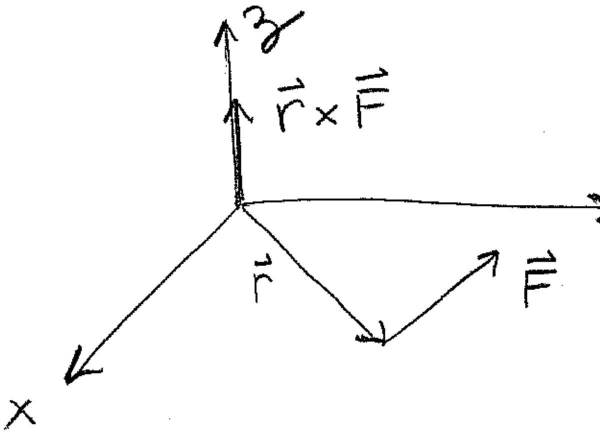
point fingers of right hand in $\vec{v}$ - direction then <u>cross</u> those fingers into $\vec{w}$-direction

Your thumb is force to point in direction roughly $\perp$ to both $\vec{v}$ and $\vec{w}$.

The precisely $\perp$ unit vector to $\vec{v}$ and $\vec{w}$ which points in same sort of direction as your thumb is $\hat{n}$.

Comment: $\vec{v} \times \vec{w}$ will be <u>normal</u> to the plane containing $\vec{v}$ and $\vec{w}$ (unless $\vec{v} = k\vec{w}$ in which case $\vec{v} \times \vec{w} = 0$)

E3 Suppose we apply $\vec{F}$ at position vector $\vec{r}$ (radial arm) and wish to find the torque about the origin,

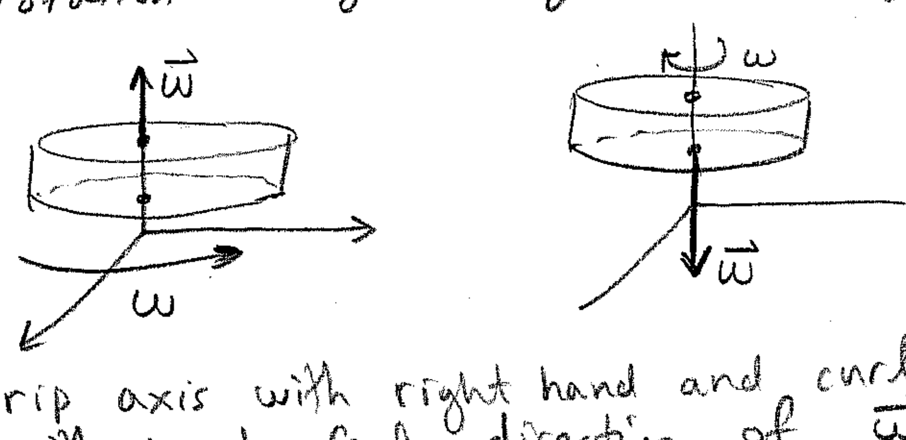


($\vec{r}$ & $\vec{F}$ both in xy-plane in this picture)

We expect $\vec{F}$ produces a rotation with axis the z-axis. Notice the cross-product of $\vec{r}$ & $\vec{F}$ points along z-axis

Def^{n}/ $\vec{\tau} = \vec{r} \times \vec{F}$ gives torque of $\vec{F}$ relative to origin from which $\vec{r}$ originates.

<u>Convention</u>: we replace ω, α with $\vec{\omega}$ and $\vec{\alpha}$ in this context. The direction for $\vec{\omega}$ and $\vec{\alpha}$ points along axis and the positive sense of rotation is given by another right-hand-rule



(I chose rotating cylinders for illustrational purposes mostly)

(grip axis with right hand and curl fingers with ω to find direction of $\vec{\omega}$)

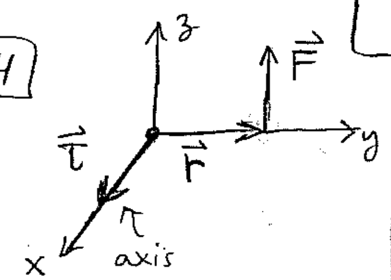
Vector form of rotational dynamical eq$^{\underline{n}}$s

$$\vec{\tau}_{net} = \sum_j \vec{r} \times \vec{F}_j$$

$$\vec{\tau}_{net} = I\vec{\alpha} \quad \text{and} \quad \vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

The direction of these angular quantities describes the axis of the rotation.

E4

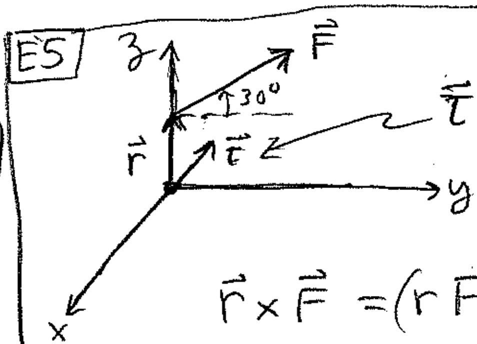


$\vec{r} = r\hat{j}$

$\vec{F} = F\hat{k}$

$\vec{r}\times\vec{F} = rF\,\hat{j}\times\hat{k} = rF\,\hat{i}$

E5



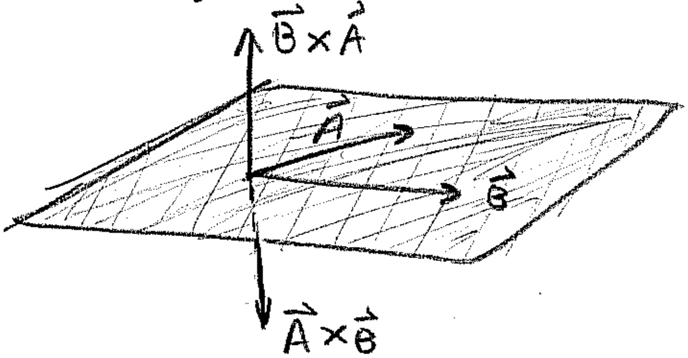
$\vec{\tau}$ points into page

$$\vec{r}\times\vec{F} = (rF\sin 60^\circ)\,\hat{n} = \frac{\sqrt{3}\,rF}{2}(-\hat{i})$$

<u>Key Idea</u>: $\vec{A}\times\vec{B}$ is $\perp$ to both $\vec{A}$ and $\vec{B}$. This is most of what the RHR does for us.

$$(\vec{A}\times\vec{B})\cdot\vec{A} = 0 \quad \underline{\text{AND}} \quad (\vec{A}\times\vec{B})\cdot\vec{B} = 0$$

Slight ambiguity, there are two choices, above or below the plane

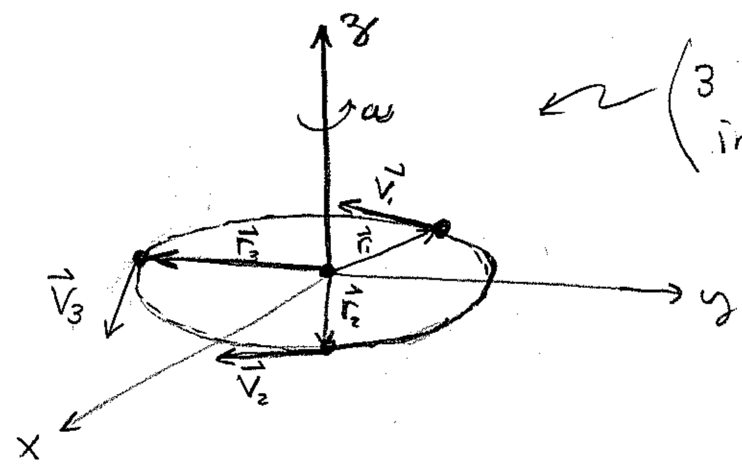


the RHR picks a side once we fix $\vec{A}$, $\vec{B}$.

Angular Momentum

Following Tipler, we imagine a mass m constrained to rotate at ω in a circle of radius r about an axis (say z for convenience)

$\boxed{E6}$



(3 different points in the orbit shown.)

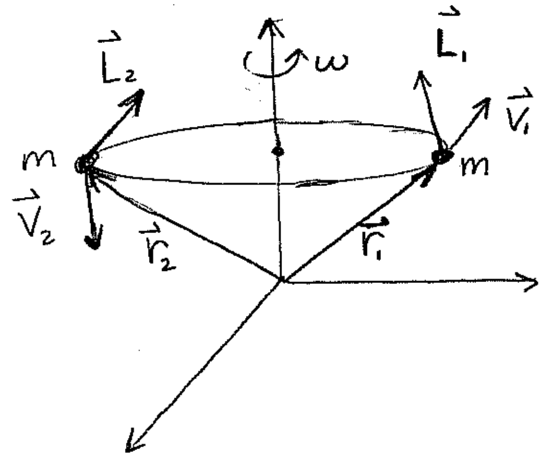
Notice $\vec{r}_1 \times \vec{V}_1$, $\vec{r}_2 \times \vec{V}_2$ and $\vec{r}_3 \times \vec{V}_3$ all point in direction of positive z-axis (a.k.a $\hat{k}$).
It's clear that $\vec{r} \times \vec{V}$ will point in the direction of the axis if the motion is a rotation. The tendency of a particle to remain in rotational motion is naturally described by <u>ANGULAR MOMENTUM</u>

$$\vec{L} = \vec{r} \times \vec{p}$$
$$= \vec{r} \times (m\vec{v})$$
for $\boxed{E6}$
$$= m\vec{r} \times \vec{v}$$
$$= (mrv)\,\hat{k}$$

(angular momentum w.r.t. particular origin, also $\vec{p} = m\vec{v}$ is <u>linear momentum</u> or just momentum)

We see $\vec{L}$ points in direction of rotation axis for the figure above. We'd like to understand how $\vec{L}$ changes and works for other cases. Next we consider rotation where the origin is not in orbital plane.

Next analyze motion where $\vec{r}$ is not in plane of motion.

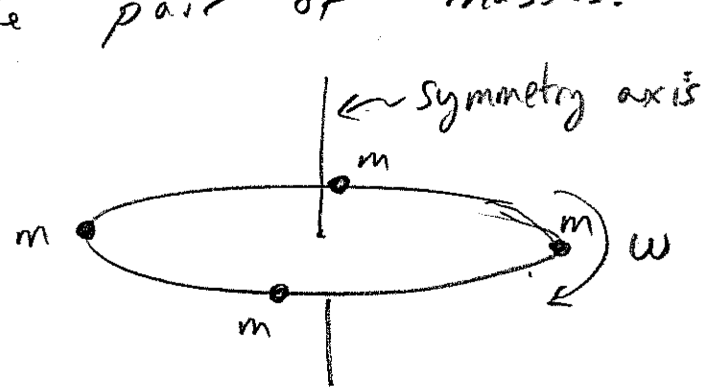


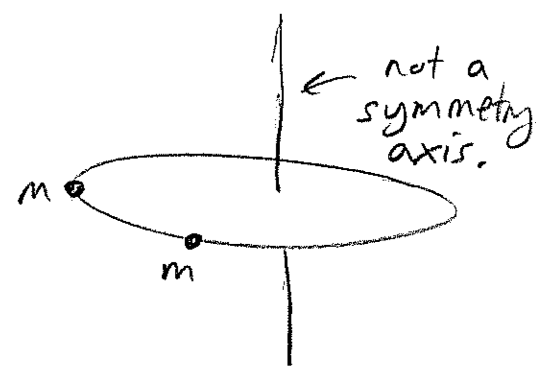
Now $\vec{L}$ changes direction as we examine different positions around the path.

(see figure 10-13 on p. 335)
notice how he draws $\vec{L}_1$ and $\vec{L}_2$ based at the point where $\vec{r}$ is based. I'm thinking of $\vec{L}_1$ & $\vec{L}_2$ as snapshots for a single mass.
Of course, if we have a pair of masses then $\vec{L}_1 + \vec{L}_2 = \vec{L}_{TOTAL}$ and if the masses move in circle with m & m diametrically opposite then

$$\vec{L}_{TOTAL} = \vec{L}_1 + \vec{L}_2 = L_{TOTAL}\,\hat{k}$$

This makes z a symmetry axis for the pair of masses.





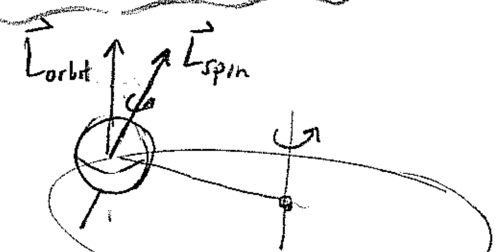
$$\boxed{\text{Th}^{m}/ \quad \vec{\tau}_{\text{net ext.}} = \frac{d\vec{L}}{dt}}$$

Proof: for single particle to keep it simple here,

$$\frac{d}{dt}\left(\vec{L}\right) = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right)$$
$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$
$$= \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$= \vec{\tau}.$$

For a system we basically end up adding this argument over the constituent masses.

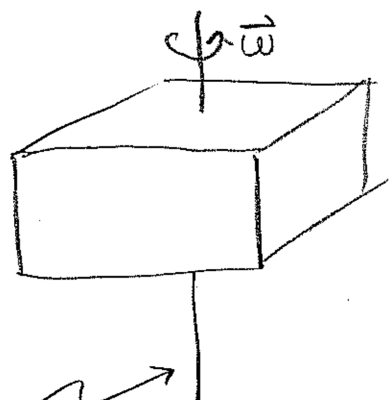
CONCEPT : If $\vec{\tau}_{net} = 0$ then $\vec{L}_i = \vec{L}_f$ (angular momentum is conserved)



- watch gyroscope video.
- Discuss $\vec{L}_{sys} = \vec{L}_{orbit} + \vec{L}_{spin}$
- work out Example 10-3.

Symmetry Axis Considerations

When $\vec{L}$ is in same direction as a symmetry axis for a system rotating at $\vec{\omega}$ around the axis we have $\vec{L} = I\vec{\omega}$. The relation between $\vec{L}$ and the net torque applied to the system $\vec{\tau}$ is



axis of symmetry passes through the com for the body

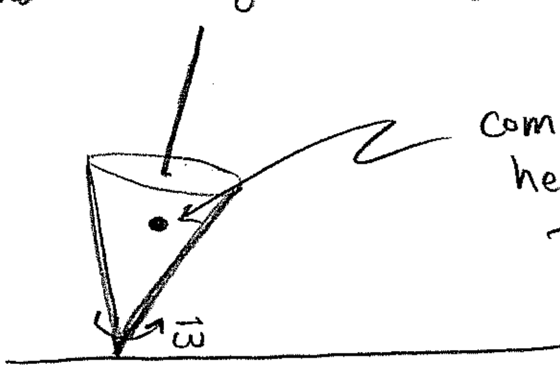
$$\boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}}$$

← rotation analogue of $\vec{F}_{net} = \frac{d\vec{P}}{dt}$

This eq^{n} says that

$$\boxed{d\vec{L} = \vec{\tau}_{net}\, dt}$$

Think about this, if $\vec{\tau}_{net} = 0$ then $d\vec{L} = 0$ hence $\vec{L}_i = \vec{L}_f$. It follows that $I\vec{\omega} = \text{Constant}$ so the body keeps spinning without wobbling. On the other hand if $\vec{\tau} \neq 0$ is applied and $\vec{\tau} \nparallel \vec{\omega}$ then the direction of $\vec{\omega}$ will change and the system may wobble.



com, gravity pulls down here producing a torque. This causes the top to change its rotation axis and it wobbles and precesses till it falls down.

Remark: to find symmetry axes need inertia tensor (see 321 notes)