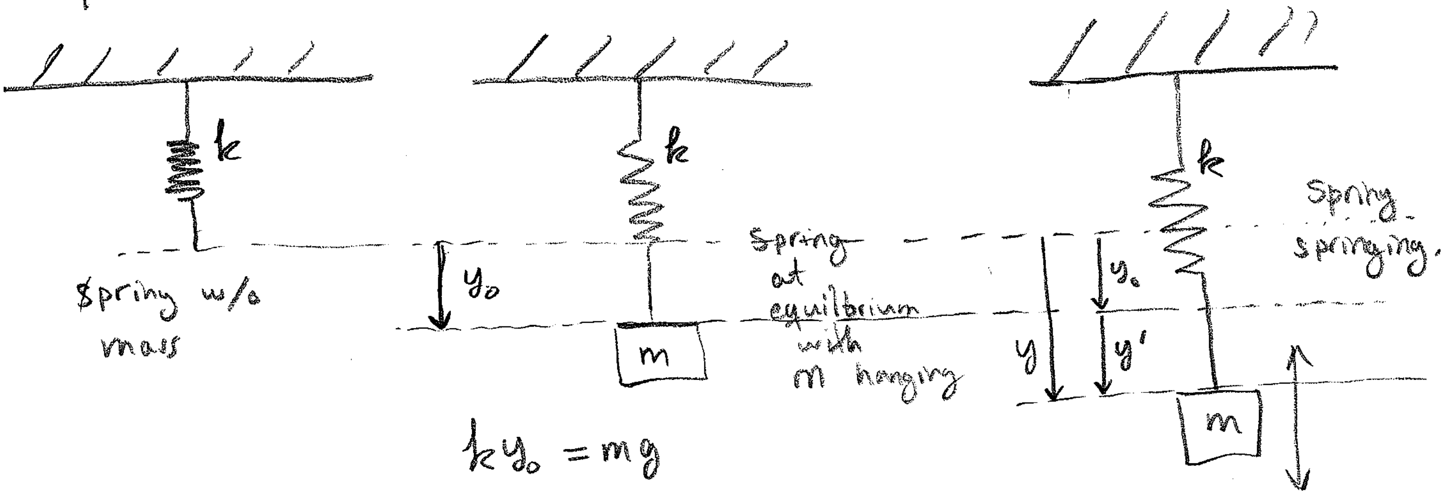
LECTURE 37

- Additional Examples of SHM and Analysis of Damped Harmonic Motion. (we continue to assume the mathematical background covered in Lecture 36)

E1 Hang a spring vertically and analyze motion

3 pictures to consider.



$$ky_o = mg$$

$$\therefore \underline{y_o = \frac{mg}{k}}$$

$$y = y_o + y'$$

Forces on m: $-ky + mg = F_{net}$ (down is positive

Newton's 2nd Law: $m\ddot{y} = -ky + mg$

$$= -ky + ky_o \quad \text{since } y_o = \frac{mg}{k}$$

$$= -k(y - y_o)$$

$$= -ky'$$

But, $y' = y - y_o \Rightarrow \ddot{y}' = \ddot{y}$ hence we find SHM in the y'-coordinate.

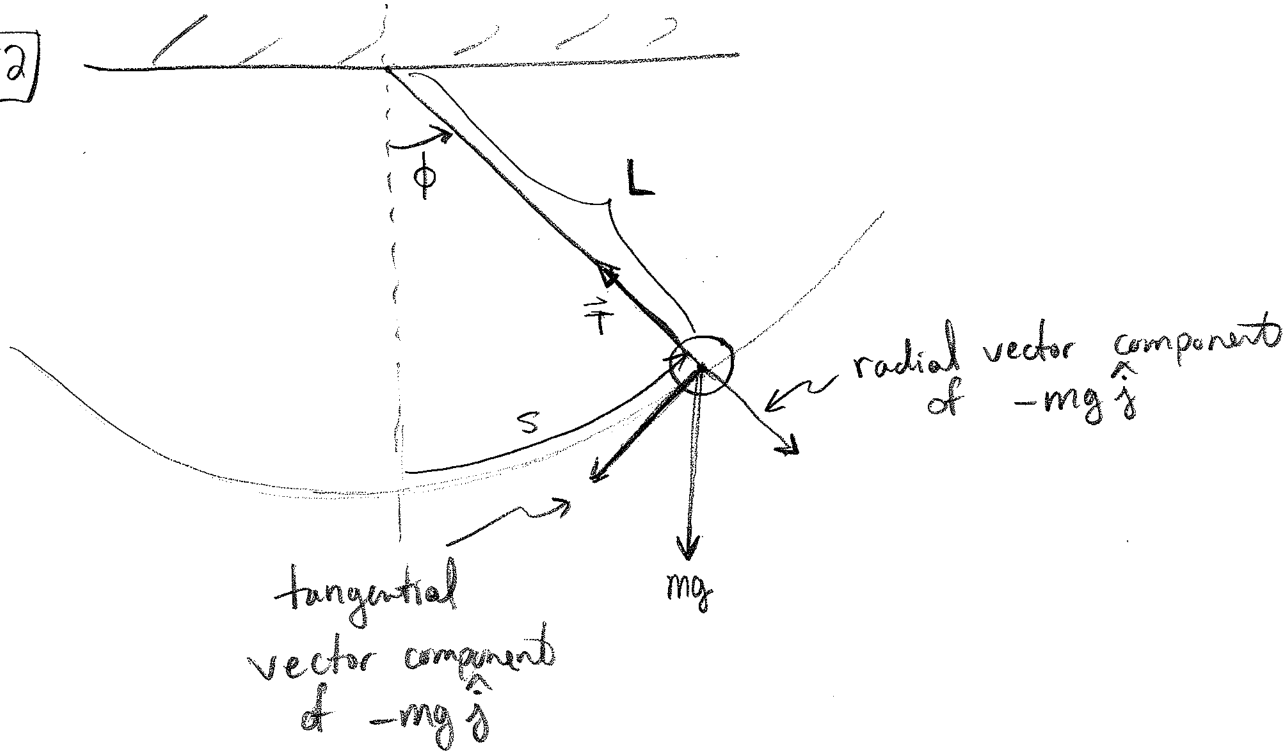
$$m\frac{d^2y'}{dt^2} + ky' = 0$$

$$\hookrightarrow \boxed{y'(t) = A\cos\left(\sqrt{\tfrac{k}{m}}\,t + \delta\right)}$$

OR $$\boxed{y(t) = y_o + A\cos\left(\sqrt{\tfrac{k}{m}}\,t + \delta\right)}$$

Simple Pendulum

E2



Newton's 2nd Law (tangential component): $m\frac{d^2s}{dt^2} = -mg\sin\phi$

But, $s = L\phi$ hence $\ddot{s} = L\ddot{\phi} \Rightarrow mL\ddot{\phi} = -mg\sin\phi$

$$\therefore \boxed{\ddot{\phi} + \frac{g}{L}\sin\phi = 0}$$

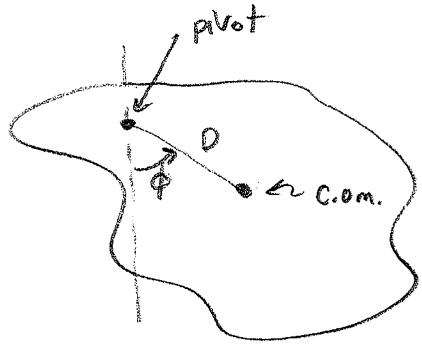
If $\phi \approx 0$ then $\sin\phi \approx \phi$ hence $\ddot{\phi} + \frac{g}{L}\phi = 0$

and we have SHM with $\omega = \sqrt{\frac{g}{L}} = \frac{2\pi}{T}$

thus $\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$ and $\phi = \phi_0 \cos(\omega t + \delta)$.

(this is an approximate sol^{n}, of course $\sin\phi \neq \phi$ so there is an error which will accumulate as time evolves. See eq^{n} 14-30 of Tipler. Hmm... wonder how to derive that!)

E3 The Physical Pendulum



for $\phi \approx 0$

$\sin\phi \approx \phi$ and

$$\tau = -(Mg\sin\phi)D \approx -MgD\phi$$

$$\Rightarrow I\alpha = -MgD\phi$$

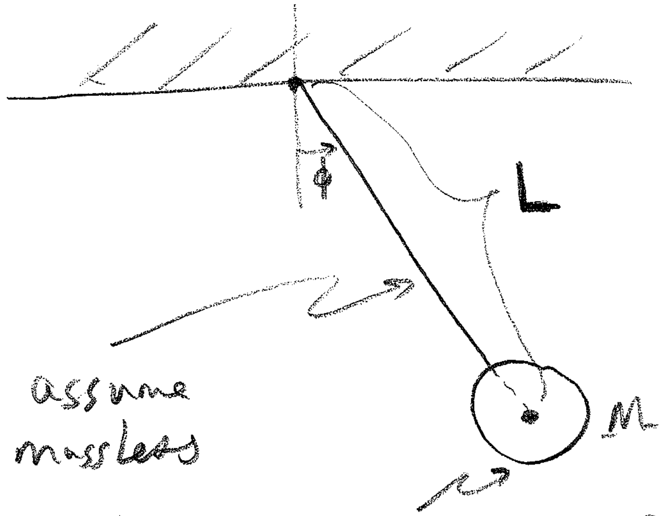
$$\Rightarrow I\ddot{\phi} = -MgD\phi$$

$$\Rightarrow \ddot{\phi} + \frac{MgD}{I}\phi = 0$$

$$\hookrightarrow \omega = \sqrt{\frac{MgD}{I}} = \frac{2\pi}{T}$$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{I}{MgD}}}$$

E4 Nice application of E3 is



assume massless

$D = L$ here.

(distance of com from pivot.

$I_{cm} = \frac{1}{2}MR^2$

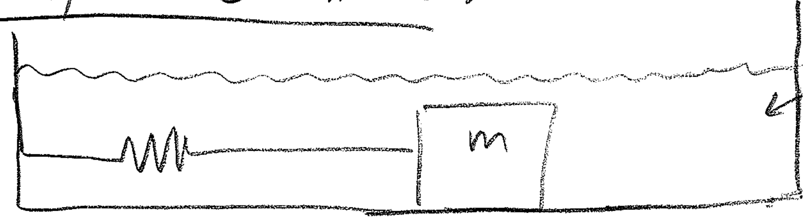
$$T = 2\pi\sqrt{\frac{\frac{1}{2}MR^2 + ML^2}{MgL}}$$

$$= 2\pi\sqrt{\frac{\frac{1}{2}R^2 + L^2}{gL}}$$

$$= \underbrace{2\pi\sqrt{\frac{1}{2}\frac{R^2}{gL} + \frac{L}{g}}}$$

can you show this $\longrightarrow \sqrt{\frac{g}{L}}$ for $R \ll L$?

Damped Oscillations:



jello. or whatever anyway

$$F_{net} = -kx - bv = ma \qquad F_f = -bv$$

where b is characteristic of the retarding media.

But, $\dot{x} = v$ and $a = \ddot{x}$ thus

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$$

$$\left(\lambda + \frac{b}{2m}\right)^2 + \frac{k}{m} - \frac{b^2}{4m} = 0$$

$$\left(\lambda + \frac{b}{2m}\right)^2 = \frac{b^2 - 4k}{4m} = -\left(\frac{4k - b^2}{4m}\right)$$

$$\lambda + \frac{b}{2m} = \pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m}}$$

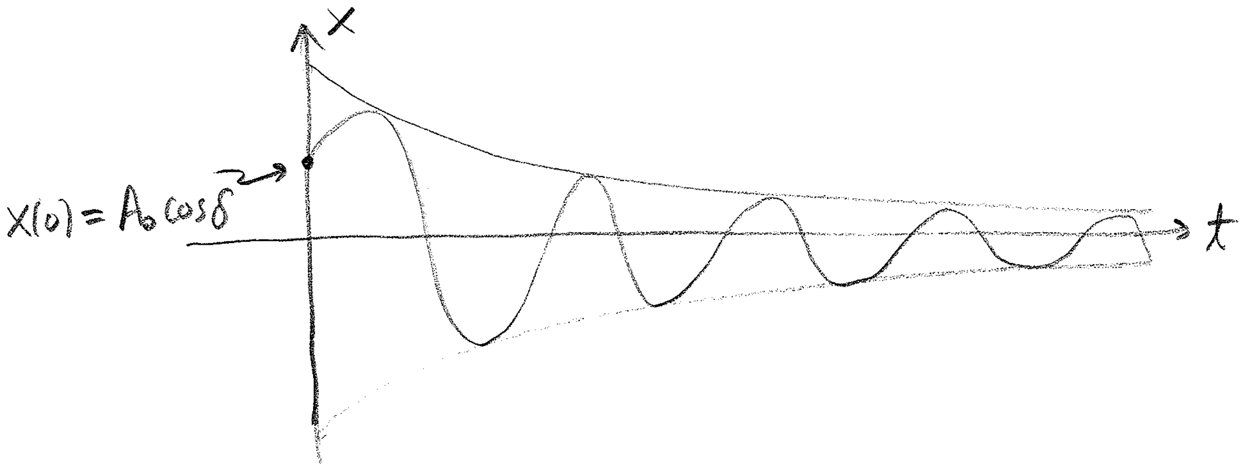
Let $\omega_0 = \sqrt{\frac{k}{m}}$ then $\omega_0^2 = \frac{k}{m}$ and

$$\lambda = \frac{-b}{2m} \pm i\sqrt{\omega_0^2\left(1 - \frac{b^2}{4m\omega_0^2}\right)}$$

$$\therefore \quad \lambda = \frac{-b}{2m} \pm i\underbrace{\omega_0\sqrt{1 - \left(\frac{b}{2m\omega_0}\right)^2}}_{\omega'}$$

$$\therefore \quad \boxed{x(t) = A_0 e^{\frac{-bt}{2m}} \cos(\omega' t + \delta)}$$

The sol$^{\underline{n}}$ $x(t) = A_0 e^{-\frac{b}{2m}t} \cos(\omega' t + \delta)$ is a cosine enveloped by $\pm A_0 e^{-bt/2m}$

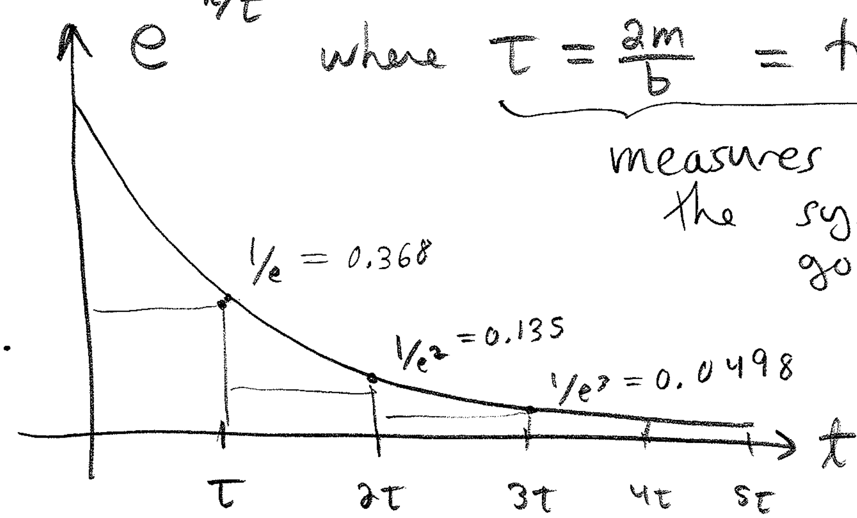


If the oscillation is fast enough then over a short enough time-scale this again looks like SHM with

$$x(t) = A\cos(\omega' t + \delta)$$

where $A = A_0 e^{-bt/2m}$ (strictly speaking this is not SHM because $\dot{A} \neq 0$)

$e^{-t/\tau}$ where $\underbrace{\tau = \frac{2m}{b} = \text{time constant}}_{\text{measures how fast the system's amplitude goes to zero.}}$



$$\frac{1}{e^5} = 0.00674$$

by 5τ the system's amplitude has decreased to under 1% of its starting value.

Def^{n}/ Quality Factor $Q = \omega_0 \tau$
describes the energy loss per cycle.