LECTURE 4

- In this lecture we study a number of interesting kinematic word problems and we catalogue the standard formulas for constant acceleration one-dimensional motion.

E1 A certain race car travels at 200 mph as it races down the straight portion of the race track. How many ft does the car travel in 3 seconds?

We assume <u>constant</u> velocity motion thus $V = V_{avg} = \frac{\Delta x}{\Delta t}$

$$
\begin{aligned}
\Delta x &= V\,\Delta t \\
&= \left(200\ \frac{\text{miles}}{\text{hr.}}\right)(3\text{ s}) \\
&= (600\cancel{s})\left(\frac{\cancel{\text{miles}}}{\cancel{\text{hr.}}}\right)\left(\frac{5280\text{ ft}}{\cancel{\text{mile}}}\right)\left(\frac{\cancel{\text{hr.}}}{3600\cancel{s}}\right) \\
&= \boxed{880\text{ ft}}.
\end{aligned}
$$

E2 Two friends, BOB and HILBERT, live in ZOUNDSVILLE. They both plan to travel out of town on interstate 30. The only rest-stop is 130 miles away from town. Suppose BOB leaves at 1:00pm and travels at an average speed of 65 mph. If HILBERT leaves at 1:15 pm then what speed does he need to drive to catch BOB at the rest-stop?

We assume constant speed for BOB and HILBERT thus $\Delta x_{BOB} = V_{BOB}\,\Delta t_{BOB}$ and $\Delta x_{HILBERT} = V_{HILBERT}\,\Delta t_{HILBERT}$. We wish to find $V_{HILBERT}$ such that $\Delta x_{BOB} = \Delta x_{HILBERT}$.

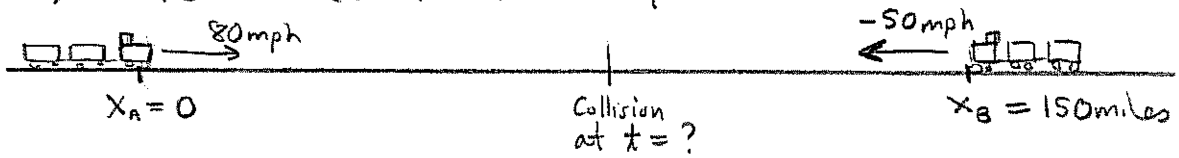
Notice $\Delta x_{BOB} = 130 \text{ miles} = (65\text{ mph})\,\Delta t_B \Rightarrow \underline{\Delta t_B = 2\text{ hrs}}$.

Thus $\Delta t_H = 1.75$ hrs (since Hilbert leaves at 1:15 pm).

Solve $\Delta x_H = V_H\,\Delta t_H$ for $V_H = \frac{\Delta x_H}{\Delta t_H} = \frac{130\text{ miles}}{1.75\text{ hrs}} = \boxed{74.29\text{ mph}}$

E3 Suppose superman is a little slow due to excessively cloudy weather shielding the warmth of our yellow Sun. He flies back and forth at a speed of 500mph between trains on a collision course. If the trains have speeds of 80 mph and 50 mph and begin 150 miles apart then how far does superman travel as he flies back & forth between trains? Assume the time for superman to reverse directions is neglible. (he just stops the trains as they almost collide since a reality TV show needs a shot for commercials that week.)

To begin let's find when the trains collide, call the trains (A) and (B) with $V_A = 80\text{mph}$ and $V_B = -50\text{mph}$. Furthermore, $X_A(0) = 0$ whereas $X_B(0) = 150$ miles. Here's a picture



To find the time of this potential collision we look for t such that $X_A(t) = X_B(t)$. Since (A), (B) move with constant velocity, (take $t = 0$ at start)

$$X_A(t) = (80\,\text{mph})\,t$$

$$X_B(t) = 150\,\text{miles} - (50\,\text{mph})\,t$$

Note $V_A(t) = 80\,\text{mph}$ and $V_B(t) = -50\,\text{mph}$ and $X_A(0) = 0$ and $X_B(0) = 150\,\text{miles}$ as the ought from the given data. Thus the collision t must solve;

$$(80\,\text{mph})\,t = 150\,\text{miles} - (50\,\text{mph})\,t$$

$$\Rightarrow t = \frac{150\,\text{miles}}{130\,\text{mph}} = \underline{1.154\ \text{hrs}}.$$

Thus we find superman flies at 500mph for 1.154 hrs giving a total of $\Delta X_{\text{superman}} = \left(500\,\frac{\text{miles}}{\text{hr}}\right)(1.154\,\text{hr}) = \boxed{577\,\text{miles}}$

The constant acceleration problem

Suppose a particle undergoes one-dimensional motion at constant acceleration a. Furthermore, suppose x_o and v_o denote the initial position and velocity of the particle. Let's use calculus to find $v(t)$ and $x(t)$.

$$a(t) = \frac{dv}{dt} \Rightarrow \int_{v_o}^{v(t)} dv = \int_0^t a\, d\bar{t}$$

since a constant.

$$\Rightarrow v(t) - v_o = at$$

$$\Rightarrow \boxed{v(t) = v_o + at}$$

Next, to find position we integrate once more,

$$v(t) = \frac{dx}{dt} \Rightarrow \int_{x_o}^{x(t)} dx = \int_0^t v(\bar{t})\, d\bar{t}$$

$$\Rightarrow x(t) - x_o = \int_0^t (v_o + a\bar{t})\, d\bar{t} = v_o t + \tfrac{1}{2}at^2$$

$$\Rightarrow \boxed{x(t) = x_o + v_o t + \tfrac{1}{2}at^2}$$

The formulas above are intended for motion which begins at time zero. Often we are confronted with problems that do not directly involve time. For one-dimensional motion we can utilize the chain-rule to find an interesting formula for velocity and position

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

this line assumes it is possible to write velocity as a function of position below. For higher-dimensional motion, not so simple.

$$\Rightarrow a\,dx = v\,dv$$

$$\Rightarrow \int_{x_o}^{x_f} a\,dx = \int_{v_o}^{v_f} v\,dv$$

$$\Rightarrow a(x_f - x_o) = \tfrac{1}{2}(v_f^2 - v_o^2)$$

$$\Rightarrow \boxed{v_f^2 = v_o^2 + 2a(x_f - x_o)}$$

(Sometimes I use $x_f - x_o = \Delta x$)

Remark: the identity $a = v\frac{dv}{dx}$ is useful for cases where a is non-constant. This is a feature of one-dim'l motion with $v \neq 0$.

E4) Suppose a car has brakes which apply a constant deceleration of $a = -2.0\,m/s^2$. If the car has an initial speed $V_0 = 20\,m/s$ then what is the stopping distance?

If the car stops then $V_f = 0$.

$$0^2 = V_0^2 + 2a(\Delta X)$$

$$\Rightarrow \Delta X = \frac{-V_0^2}{2a} = \frac{-(20\,\frac{m}{s})^2}{2(-2.0\,\frac{m}{s^2})} = \frac{400\,m}{4} = \boxed{100\,m}$$

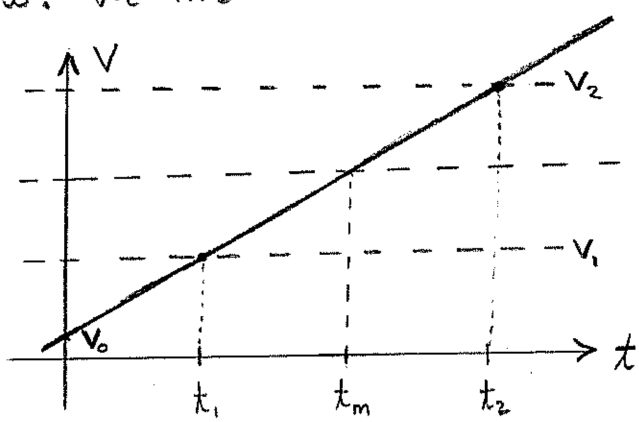
E5) How fast did the car in E4 stop?

$$V_f = V_0 + at \Rightarrow t = \frac{V_f - V_0}{a} = \frac{-20\,m/s}{-2\,m/s^2} = \boxed{10\,s}$$

E6) What was the average velocity of the car in E4 as it was braking?

$$V_{avg} = \frac{\Delta X}{\Delta t} = \frac{100\,m}{10\,s} = \boxed{10\,\frac{m}{s}}$$

Remark: $V_{avg} = \frac{1}{2}V_0$, is this an accident? Consider the graph below. We find another sometimes useful fact:



$$\boxed{V_{avg} = \frac{1}{2}(V_1 + V_2)}$$

See derivation below graph

$$V_{avg} = \frac{\Delta X}{\Delta t} = \frac{X(t_2) - X(t_1)}{t_2 - t_1} = \frac{X_0 + V_0 t_2 + \frac{1}{2}at_2^2 - X_0 - V_0 t_1 - \frac{1}{2}at_1^2}{t_2 - t_1}$$

$$= \frac{V_0(t_2 - t_1) + \frac{1}{2}a(t_2^2 - t_1^2)}{t_2 - t_1}$$

$$= V_0 + \frac{1}{2}a(t_1 + t_2) = \frac{1}{2}(\underbrace{V_0 + at_1}_{V_1} + \underbrace{V_0 + at_2}_{V_2}).$$

- **CONCEPTUAL CHECKPOINT**

  - x denotes position
  - $v = \frac{dx}{dt}$ denotes velocity which is the rate of change in position.
  - $a = \frac{dv}{dt}$ denotes acceleration which is the rate of change in velocity.

It's worth noting that the sign of x says nothing for v*
the sign of v says nothing for a*

* (that is w/o other additional information)

E7) If $x(t) = c_0 t$ where c_0 is a positive constant then $v(t) = c_0$. Notice $x < 0$ and $x > 0$ both give same velocity. The sign of position need not say much about v.

E8) If $v(t) = c_1 t$ where c_1 is a positive constant then $a(t) = c_1$. Again note the sign of v says nothing about a.

Note v and a describe the change in x or v respective. The particular values of x and v at a single time do not reveal how x and v are changing near that time. We need additional information. My point? You need to think and understand the def's and calculus. Conceptual examples also helpful

E9) A car going 50 mph could be accelerating or decelerating.

E10) A car at the 110 mile marker could be travelling towards the 109 mile marker (say $v < 0$) or towards the 111 mile marker ($v > 0$)

CALCULUS GUIDE

$$\begin{tikzcd} x \arrow[r, bend left, "d/dt"] & v \arrow[l, bend left, "\int"', "\text{given } x(t_0)"] \arrow[r, bend left, "d/dt"] & a \arrow[l, bend left, "\int"', "\text{given } v(t_0)"] \end{tikzcd}$$

And: we can use $a = v\frac{dv}{dx}$ to find t-independent formula.

$\boxed{E11}$ What if the acceleration is linearly increasing with time such that $a(t) = a_o + tj_o$ where $[j_o] = \frac{m}{s^3}$. Find the equations of motion given that $V(0) = V_o$ and $X(0) = X_o$

Note $\frac{dV}{dt} = a_o + tj_o$ where a_o, j_o are constants

Integration yields $\boxed{V(t) = V_o + a_o t + \frac{1}{2} j_o t^2}$ Then

$V = \frac{dx}{dt}$ so integration again yields $\boxed{X(t) = X_o + V_o t + \frac{1}{2} a_o t^2 + \frac{1}{6} j_o t^3}$

Finally, for fun, lets find formula connecting ΔV and ΔX directly,

$$a = V\frac{dV}{dx} \Rightarrow a_o + tj_o = V\frac{dV}{dx}$$

need to write in terms of X, V to proceed. I'll assume $a_o = 0$ to simplify task and leave $a_o \neq 0$ to the interested reader.

Special case $a_o = 0$) $V = V_o + \frac{1}{2} j_o t^2 \Rightarrow \frac{2(V - V_o)}{j_o} = t^2$

Thus $t = \pm\sqrt{\frac{2(V-V_o)}{j_o}}$. Suppose (+) applies to the given problem to find,

$$a_o + tj_o = V\frac{dV}{dx} \Longrightarrow j_o\sqrt{\frac{2(V-V_o)}{j_o}} = V\frac{dV}{dx}$$

$$\Rightarrow \int_{X_o}^{X_f} \sqrt{2j_o}\, dx = \int_{V_o}^{V_f} \frac{V\,dV}{\sqrt{V - V_o}}$$

$u = V - V_o$
$V = u + V_o$
$du = dV$
$u(V_f) = V_f - V_o$
$u(V_o) = 0$

$$\Rightarrow \sqrt{2j_o}\,[X_f - X_o] = \int_0^{V_f - V_o} \left(\frac{u + V_o}{\sqrt{u}}\right) du$$

$$\Rightarrow \sqrt{2j_o}\,[X_f - X_o] = \left(\frac{2}{3}u^{3/2} + 2V_o\sqrt{u}\right)\Big|_0^{V_f - V_o}$$

$$\Rightarrow \boxed{X_f = X_o + \frac{1}{\sqrt{2j_o}}\left[\frac{2}{3}(V_f - V_o)^{3/2} + 2V_o\sqrt{V_f - V_o}\right]}$$

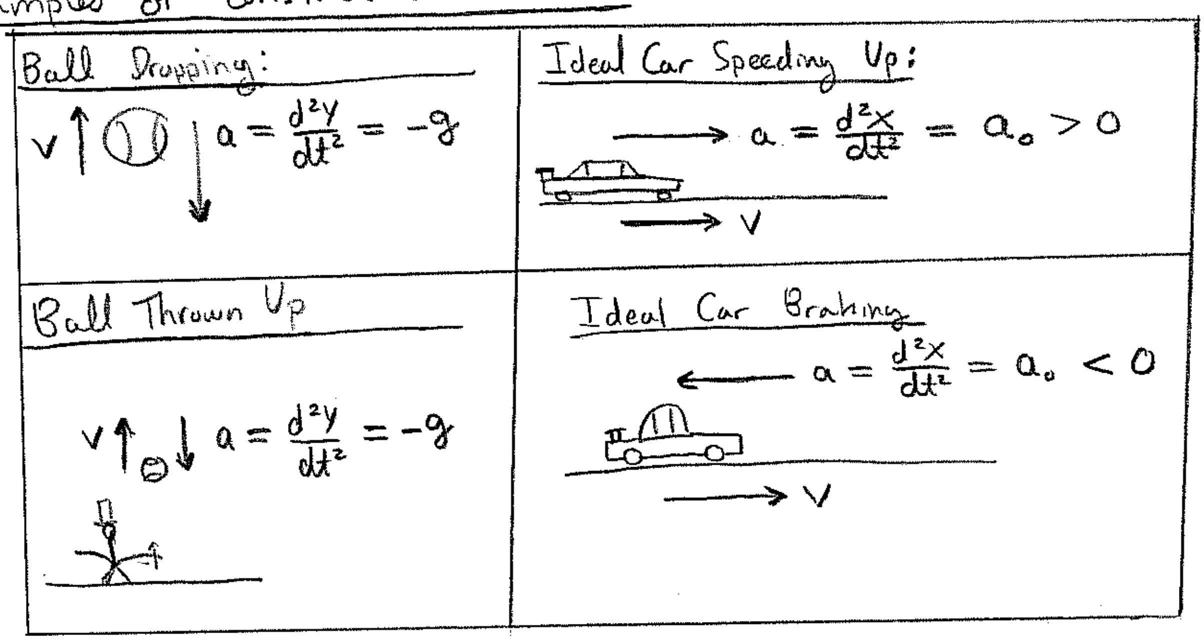
(see why we focus on <u>constant</u> acceleration case. Even linearly time-dependent a gives quite ugly eq$^{\underline{n}}$s of motion.)

Remark: E11 explores the mathematics of a nonconstant acceleration. Clearly the eq^{n}s are not as nice as

$$\boxed{V_f = V_0 + at \;\Big|\; X_f = X_0 + V_0 t + \tfrac{1}{2}at^2 \;\Big|\; V_f^2 = V_0^2 + 2a(X_f - X_0)}$$

where V_f = final velocity and V_0 = initial velocity and t is the time elapsed between the events $(t_0, X(t_0) = X_0)$ and $(t_f, X(t_f) = X_f)$ so $t = t_f - t_0$. More important than math is the fact that a = constant is quite common in everyday physics (at least to a good approximation: and by the way all physics is approximate) ← (my opinion)

Examples of constant acceleration:



We follow § 2.3 of Tipler and examine a few examples involving the situations above and more. The strategies and formulas used apply to other constant acceleration cases. However, you must be careful to apply the constant a formulas only to the case where they apply. In two, or three, dimensional motion they still might apply. We need motion along some coordinate to be constant acceleration w.r.t. said coordinate. We'll see this for a number of lectures to follow.

Remark: $g = 9.81\,\frac{m}{s^2}$. This is our convention.

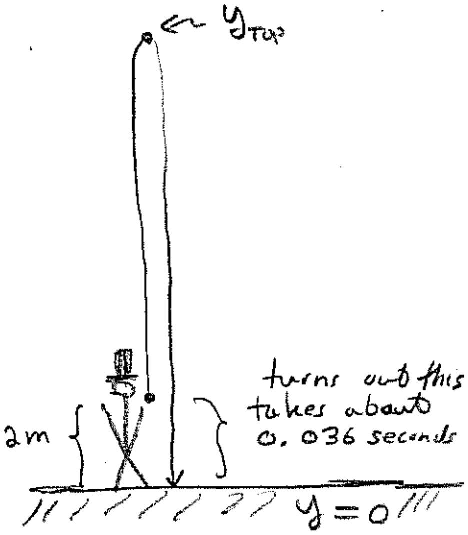
E12 If we throw a ball vertically with a speed of $V_o = 50\,\frac{m}{s}$ then what is the maximum height of the ball and how long is it in flight? Assume the ball is released from $y = 2m$ and the ground where it falls to is at $y = 0m$

Notice the top flight has $V = 0$ hence we find t_{top} by solving $V_f = V_o - gt_{top} \longrightarrow t_{top} = \frac{V_o}{g} \cong 5.097s$

Since balls (ignoring friction) fall under constant acceleration of $\frac{dV}{dt} = -g$ we place $a = -g$ in our formula from (3),

$$\begin{aligned} y_{TOP} &= y_o + V_o t - \tfrac{1}{2} g t^2 \\ &= y_o + V_o\left(\frac{V_o}{g}\right) - \frac{1}{2} g \left(\frac{V_o}{g}\right)^2 \\ &= y_o + \frac{V_o^2}{2g} \\ &= 2.0m + \frac{1}{2(9.81\,m/s^2)}\left(50\,\frac{m}{s}\right)^2 \\ &\cong \boxed{129.4m} \end{aligned}$$

← max height above ground.



Finally to find time of flight solve $y(t) = 0$ for t

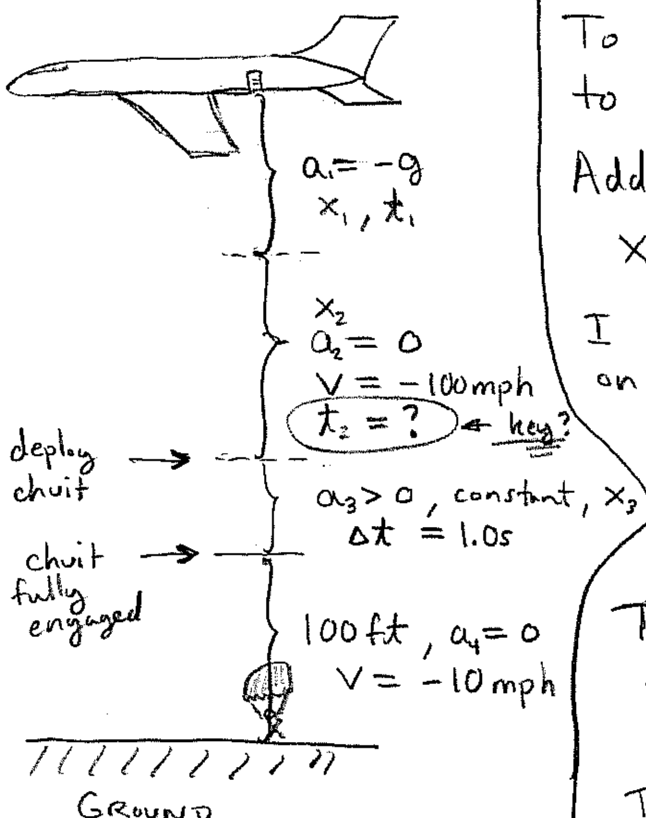
$$\begin{aligned} y(t) = 0 &= y_o + V_o t - \tfrac{1}{2} g t^2 \\ \Rightarrow 0 &= gt^2 - 2V_o t - 2y_o \\ t &= \frac{2V_o \pm \sqrt{4V_o^2 + 8gy_o}}{2g} \end{aligned}$$

(+) is clearly the physically interesting sol$^{\underline{o}}$!

$$\begin{aligned} t &= \frac{2V_o + \sqrt{4V_o^2 + 8gy_o}}{2g} \\ &= \frac{2(50\,\frac{m}{s}) + \sqrt{4(50\,\frac{m}{s})^2 + 8(9.81\,\frac{m}{s^2})(2.0m)}}{2(9.81\,\frac{m}{s^2})} \\ &\cong \boxed{10.23s} \end{aligned}$$

← time of flight.

E13 A parachuitist drops a a constant terminal velocity of 100 mph until he opens his chute and decelerates to 10mph in a time of $\Delta t = 1.0s$. A Halo jump is intendend to drop in a soldier as close to the ground as possible w/o using the ground to decelerate. Suppose he aims to have his parachuit fully deployed (meaning he's decelerated to 10mph) at 100ft. If our soldier drops vertically from 30,000 ft how long should he wait to open the parachuit? Assume constant acceleration g to the terminal velocity initially.



Conversions

$100\,mph \cong 44.74\ \frac{m}{s}$

$10\,mph \cong 4.474\ \frac{m}{s}$

To begin I draw a picture to organize the given data. Add labels as needed.

$$X_1 + X_2 + X_3 + 100ft = 30{,}000ft$$

I think we'll work backwards on this one. Note

$$a_3 = \frac{V_f - V_o}{\Delta t} = \left(\frac{90\,mph}{1.0s}\right)\left(\frac{0.44738\ m/s}{mph}\right) \cong 40.26\ m/s^2$$

Then $V_f^2 - V_o^2 = 2a_3 X_3$ hence

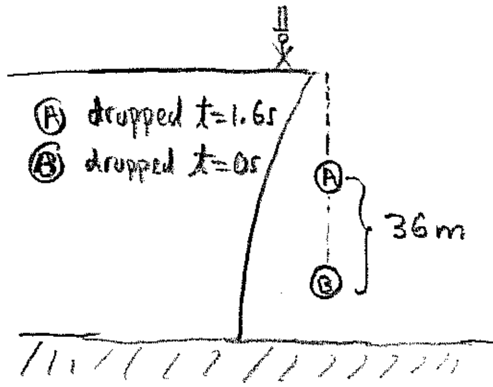
$$X_3 = \frac{(44.74\frac{m}{s})^2 - (4.474\frac{m}{s})^2}{2(40.26\ m/s^2)} = \underline{24.61\,m}$$

Thus $\underline{X_3 = 80.74ft}$. Continuing, I don't know t_2 yet so I have to think about top leg of the fall. We can find X_1 from the fact $V_f = -100mph = -44.74\frac{m}{s}$ and $V_f = V_o - gt_1 = 0 - gt_1$ we find $t_1 = \frac{-44.74\,m/s}{9.81\,m/s^2} = \underline{4.561s}$, note $X_1 = \frac{V_f^2 - V_o^2}{-2g} = 334.7ft$.

Finally $X_2 = 30{,}000ft - 100ft - 80.74ft - 334.7ft = \underline{29{,}484.6\,ft = 8986.5m}$.
Thus $t_2 = \frac{8986.5m}{44.74\,m/s} = \underline{200.9s}$. (constant velocity during terminal velocity fall). We should tell him to open it at $\boxed{t_1 + t_2 = 205.4s \text{ after jumping.}}$

E14 Two stones are dropped from a cliff, the second stone 1.6s after the first. How far below the top of the cliff is the second stone when the separation between the stones is 36 m? (#85 of p. 59 Tipler)

Both stones are dropped which means $V_o = 0\ \text{m/s}$ and both stones are falling under $g = 9.81\ \text{m/s}^2$ (ignore friction)



$$y_A = y_o - \tfrac{1}{2}g\,(t-1.6s)^2 \quad , \quad \underbrace{\Delta t = t - 1.6s}_{\text{time of flight for Ⓐ}}$$

$$y_B = y_o - \tfrac{1}{2}gt^2$$

y_o = height of cliff

whereas $\Delta t = t$ for Ⓑ.

$\underbrace{y=0}_{\text{my choice.}}$

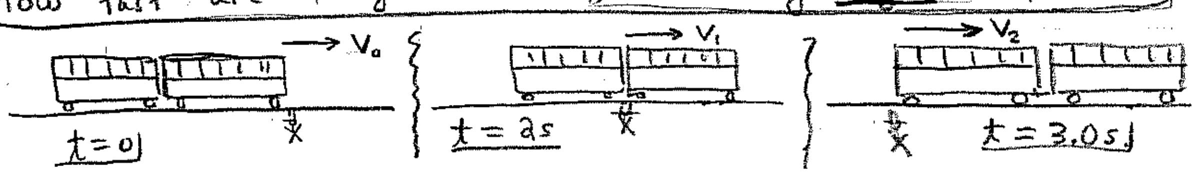
$\underrightarrow{\text{Given}}$ $y_A - y_B = 36m = -\tfrac{1}{2}g(t-1.6s)^2 + \tfrac{1}{2}gt^2$

$$\Rightarrow \frac{72m}{g} = t^2 - \left[t^2 - (3.2s)t + 2.56s^2\right]$$

$$\Rightarrow t = \frac{72m/g - 2.56s^2}{3.2s} \cong 1.4936s$$

$$\Rightarrow |y_A - y_o| = \tfrac{1}{2}gt^2 = \frac{(9.81\,\frac{m}{s^2})(1.4936s)^2}{2} = \boxed{10.94m}$$

E15 A subway car accelerates as it leaves the station. As it passes a certain person waiting for another train it takes 2.0 s for one car, and 1.0s for the next subway car to pass. If the cars have length 10m then how fast are they travelling as they begin to pass? we must find V_o.



$$V_{avg(1)} = \frac{10m}{2s} = 5\,\frac{m}{s} = \tfrac{1}{2}(V_o + V_1) \Rightarrow V_o + V_1 = 10\,\frac{m}{s}.$$

$$V_{avg(2)} = \frac{10m}{1s} = 10\,\frac{m}{s} = \tfrac{1}{2}(V_1 + V_2) \Rightarrow V_1 + V_2 = 20\,\frac{m}{s}.$$

$$V_{avg(3)} = \frac{20m}{3s} = \frac{20}{3}\,\frac{m}{s} = \tfrac{1}{2}(V_o + V_2) \Rightarrow V_o + V_2 = \frac{40}{3}\,\frac{m}{s}$$

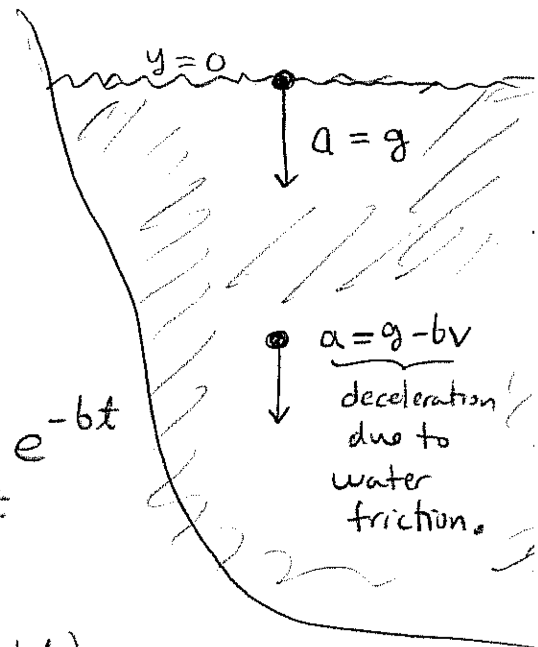
After a little algebra we find $\boxed{V_o = \frac{5}{3}\,\frac{m}{s}}$

<u>Remark</u>: $\boxed{E15}$ is equivalent to the problem of two runners who share a common acceleration and a host of other tricky problems. I leave the rest for you to discover in homework ☺.

I'll conclude with a <u>relaxing mathematical example</u>.

$\boxed{\begin{array}{l} \text{E16 / \#117 of p. 61 Tipler} \\ \text{Given } a_y = g - bV \text{ where } g, b > 0 \text{ are} \\ \text{positive constants find } a_y \text{ as a function of } \underline{\text{time}} \end{array}}$

(models friction) [acceleration as fnct. of velocity.]



$$a_y = \frac{dV}{dt} = g - bV$$

$$\Rightarrow \int \frac{dV}{g - bV} = \int dt$$

$$\Rightarrow -\frac{1}{b}\ln|g - bV| = t + c_1$$

$$\Rightarrow \ln|g - bV| = -bt - bc_1$$

$$\Rightarrow |g - bV| = e^{-bt - bc_1} = e^{-bc_1}e^{-bt}$$

$$\Rightarrow g - bV = \underbrace{\pm e^{-bc_1}}_{k} e^{-bt}$$

$$\therefore \ V(t) = \frac{1}{b}\left(g - ke^{-bt}\right)$$

$$\Rightarrow \frac{dV}{dt} = \left(-\frac{k}{b}\right)\left(-be^{-bt}\right) = ke^{-bt}$$

Hence, $a(t) = ke^{-bt}$. At time zero it is just entering the water so $a(0) = g = ke^0 = k$

$$\therefore \ \boxed{a(t) = ge^{-bt}} \text{ and } \boxed{V(t) = \frac{g}{b}\left(1 - e^{-bt}\right)}$$

Notice $V(t) \longrightarrow \frac{g}{b}$ as $t \longrightarrow \infty$ and we identify $V_\infty = g/b$ as the <u>terminal velocity</u>.