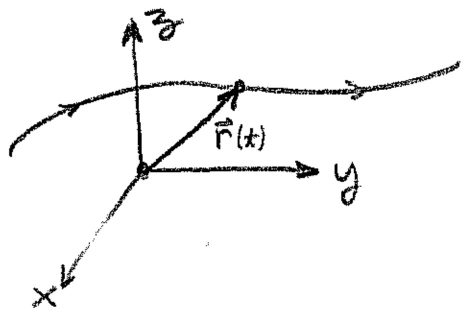
LECTURE 5

- we extend our study of kinematics to the context of several dimensions. We also study the concept of relative velocity.

Def^{n}/ $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ denotes the <u>position</u> of a particle at time t. The mapping $t \longmapsto \vec{r}(t)$ is called a path or trajectory. The pair $(t, \vec{r}(t))$ is called an event.

Notice we assume there exists an xyz-coordinate system with which we can reasonably place $\vec{r}(t)$. You can picture it with diagrams such as:

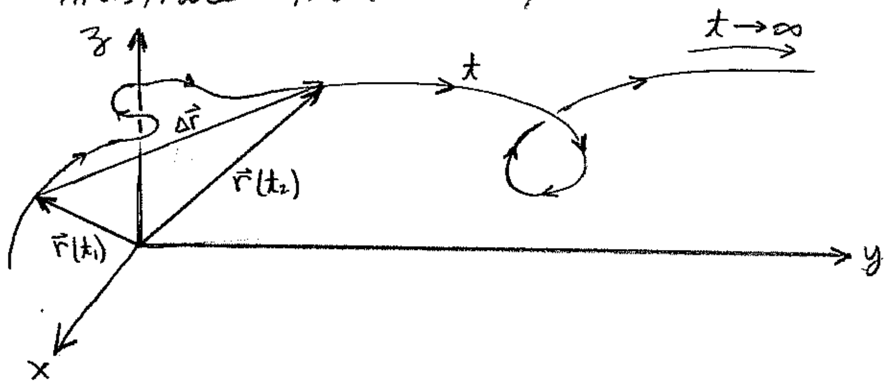


$$\underbrace{\vec{r} : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3}$$

mathematically this is a parametrized curve where the parameter is time t.

Def^{n}/ Given two events $(t_1, \vec{r}(t_1))$ and $(t_2, \vec{r}(t_2))$ with $t_1 < t_2$, we can calculate the <u>displacement</u> $\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$ and <u>average velocity</u> $\vec{V}_{avg} = \dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{1}{t_2 - t_1}\left[\vec{r}(t_2) - \vec{r}(t_1)\right]$

Let us illustrate these concepts:



E1 Suppose a particle goes and visits all amusement parks and then returns home. This $\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \vec{0}$.

I put vectors on my zero vectors. Tipler doesn't, why?

Calculus for Paths in Short:

Suppose $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is a path. Hence x, y, z are functions of t,

$$\boxed{\frac{d\vec{r}}{dt} \overset{\text{def}}{=} \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}} \qquad \boxed{\int \vec{r}\,dt \overset{\text{def}}{=} \left(\int x\,dt\right)\hat{i} + \left(\int y\,dt\right)\hat{j} + \left(\int z\,dt\right)\hat{k}}$$

Differentiation and integration of paths is <u>defined</u> componentwise. There is <u>much</u> more to say, however, this suffices for this lecture.

E2 Suppose $\vec{r}(t) = e^t\hat{i} + (\cos t)\hat{j} + t^3\hat{k}$ then calculate $d\vec{r}/dt$ and $\int \vec{r}(t)\,dt$ (I'm omitting units here)

$$\frac{d\vec{r}}{dt} = \frac{de^t}{dt}\hat{i} + \frac{d}{dt}(\cos t)\hat{j} + \frac{d}{dt}(t^3)\hat{k} = \underline{e^t\hat{i} - (\sin t)\hat{j} + 3t^2\hat{k}}.$$

$$\begin{aligned} \int \vec{r}(t)\,dt &= \hat{i}\int e^t\,dt + \hat{j}\int \cos t\,dt + \hat{k}\int t^3\,dt \\ &= (e^t + c_1)\hat{i} + (\sin t + c_2)\hat{j} + \left(\tfrac{1}{4}t^4 + c_3\right)\hat{k} \\ &= \underline{e^t\hat{i} + \sin t\,\hat{j} + \tfrac{1}{4}t^4\hat{k} + \vec{c}}. \quad (\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}). \end{aligned}$$

Def^n/ Suppose $\vec{r}(t)$ denotes the position of a particle for $t \geq t_1$ then we define velocity, speed, acceleration and distance travelled

1.) $\vec{V}(t) = \frac{d\vec{r}}{dt}$ ← velocity.
2.) $V(t) = |\vec{V}(t)| = \sqrt{\vec{V} \cdot \vec{V}}$ ← speed.
3.) $\vec{a}(t) = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2}$ ← acceleration.
4.) $s(t) = \int_{t_1}^{t} V(u)\,du$ is distance travelled during $[t_1, t]$.

Notice that s is called <u>arclength</u> in mathematics. You can note that $\frac{ds}{dt} = \frac{d}{dt}\int_{t_1}^{t} V(u)\,du = V(t) = |\vec{V}(t)|$. The speed is the magnitude of the velocity which happens to also be the rate of change in arclength w.r.t. time.

E3 Suppose $\vec{r}(t) = (R\cos \omega t)\hat{i} + (R\sin \omega t)\hat{j}$. Find distance travelled for $0 \leq t \leq 2\pi/\omega$. Assume R, ω are positive constants.

Note, $\vec{V}(t) = (-R\omega\sin \omega t)\hat{i} + (R\omega\cos \omega t)\hat{j} \Rightarrow V(t) = R\omega$

Thus $s(2\pi/\omega) = \int_0^{2\pi/\omega} R\omega\,dt = R\omega t\Big|_0^{2\pi/\omega} = \boxed{2\pi R}.$

Remark: You should recognize $\vec{r}(t) = (R\cos\omega t)\hat{i} + (R\sin\omega t)\hat{j}$ as the parametric form of a circle of radius R. Perhaps, you saw $x = R\cos(\omega t)$ and $y = R\sin\omega t$ in your course. Notice,

$$\boxed{x^2 + y^2 = R^2\cos^2(\omega t) + R^2\sin^2(\omega t) = R^2.}$$

circle radius R.

Parametrized curves sometimes challenge students in Math 132 and 231. This is a worthwhile battle since the proper context for kinematics <u>needs</u> the concept of parametric curves. In physics we have the luxury of thinking of t as time. In pure math, the parameter could be most any thing. All of this said, perhaps you've not seen parametrized curves yet in your courses. No problem. These notes are self-contained.

E4 Suppose $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ is constant $\left(\frac{d\vec{a}}{dt} = 0\right)$ and $\vec{v}(0) = V_{ox}\hat{i} + V_{oy}\hat{j} + V_{oz}\hat{k} = \vec{V}_o$ and $\vec{r}(0) = x_o\hat{i} + y_o\hat{j} + z_o\hat{k} = \vec{r}_o$ are given. Find $\vec{r}(t)$ and $\vec{v}(t)$

$$\frac{d\vec{v}}{dt} = \vec{a} \implies \vec{v}(t) = t\vec{a} + \vec{c}_1$$

calculus justified at end of this lecture

Note that $\vec{v}(0) = \vec{V}_o = \vec{c}_1$ $\therefore$ $\underbrace{\boxed{\vec{v}(t) = \vec{V}_o + t\vec{a}}}_{\text{points in direction of motion.}}$

Furthermore,

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{V}_o + t\vec{a}$$

$$\implies \vec{r}(t) = t\vec{V}_o + \tfrac{1}{2}t^2\vec{a} + \vec{c}_2$$

Apply $\vec{r}(0) = \vec{r}_o = \vec{c}_2$ $\therefore$ $\overbrace{\boxed{\vec{r}(t) = \vec{r}_o + t\vec{V}_o + \tfrac{1}{2}t^2\vec{a}}}^{\text{points from origin to where particle is.}}$

Remark: I did the calculus different this time. Previously I integrated with bounds, this approach is $\approx$.

E5 Suppose $\vec{a} = -g\hat{j}$. Assume two-dimensional context and find position, velocity and average velocity if we're given $\vec{r}(0) = x_o\hat{i} + y_o\hat{j} = \vec{r}_o$ and $\vec{V}(0) = V_{ox}\hat{i} + V_{oy}\hat{j} = \vec{V}_o$

Since $\vec{a} = \dfrac{d\vec{V}}{dt} = -g\hat{j} \Rightarrow \vec{V}(t) = -gt\hat{j} + \vec{C}_1$.

Note $\vec{V}(0) = \vec{C}_1 \Rightarrow \boxed{\vec{V}(t) = -gt\hat{j} + \vec{V}_o}$. — I

Likewise, $\vec{V} = \dfrac{d\vec{r}}{dt} = -gt\hat{j} + \vec{V}_o \Rightarrow \vec{r}(t) = -\frac{1}{2}gt^2\hat{j} + t\vec{V}_o + \vec{C}_2$.

Apply $\vec{r}(0) = \vec{C}_2 \Rightarrow \boxed{\vec{r}(t) = -\frac{1}{2}gt^2\hat{j} + t\vec{V}_o + \vec{r}_o}$ — II

Finally, average velocity from t_1 to t_2 for $t_1 < t_2$ is

$$\vec{V}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

think.

$$= \frac{1}{t_2 - t_1}\left[-\frac{1}{2}g\,(t_2^2 - t_1^2)\,\hat{j} + (t_2 - t_1)\vec{V}_o\right]$$

$$= \vec{V}_o - \frac{g}{2}\left(\frac{1}{t_2 - t_1}\right)(t_2 - t_1)(t_2 + t_1)\,\hat{j}$$

$$= \frac{1}{2}\left[\underbrace{\vec{V}_o - gt_1\hat{j}} + \underbrace{\vec{V}_o - gt_2\hat{j}}\right]$$

$$= \frac{1}{2}\left[\vec{V}_1 + \vec{V}_2\right].$$

Remark: I bet this is true for any constant acceleration motion.

Scalar Content

I →
$$\boxed{\begin{aligned} V_x &= V_{ox} \\ V_y &= V_{oy} - gt \end{aligned}}$$

II →
$$\boxed{\begin{aligned} x &= x_o + tV_{ox} \\ y &= y_o + tV_{oy} - \tfrac{1}{2}gt^2 \end{aligned}}$$

(for word problems, often the scalar eq^{n}'s are what we need. However, the vector viewpt. has many advantages.)

Remark: $\boxed{E5}$ provides a basis for a wide variety of projectile motion problems. We discuss those in the next Lecture.

Def^{n}/ Average acceleration. Let $\vec{r}(t)$ denote position of some particle and suppose $t_1 < t_2$ then $\vec{a}_{avg}$ with respect to the time interval $[t_1, t_2]$ is

$$\vec{a}_{avg} = \frac{\Delta \vec{V}}{\Delta t} = \frac{\vec{V}(t_2) - \vec{V}(t_1)}{t_2 - t_1}$$

$\boxed{E6}$ Let $\vec{r}(t) = R_1 e^{at}\hat{i} - R_2 \sin(bt+\phi)(\hat{j} - \hat{k})$
What dimensions must be given to R_1, R_2, a, b, ϕ to make this a physically reasonable position function?
Find $\vec{a}_{avg}$ for $0 \leq t \leq 1s$.

Clearly R_1, R_2 must have dimensions of length.
Arguments for exp and sin must be dimensionless thus $[at] = 1$ and $[bt] = 1 \Rightarrow [a] = [b] = \frac{1}{[t]}$.
The constant ϕ should also be dimensionless.

Differentiate $\vec{r}(t)$ to calculate the velocity,

$$\begin{aligned}
\vec{V}(t) &= \frac{d}{dt}\Big[R_1 e^{at}\hat{i} - [R_2 \sin(bt+\phi)](\hat{j} - \hat{k})\Big] \\
&= R_1 \hat{i} \frac{d}{dt}\left(e^{at}\right) - R_2(\hat{j} - \hat{k})\frac{d}{dt}\left(\sin(bt+\phi)\right) \\
&= aR_1 e^{at}\hat{i} - [bR_2 \cos(bt+\phi)](\hat{j} - \hat{k})
\end{aligned}$$

pulled out constant scalars and vectors.

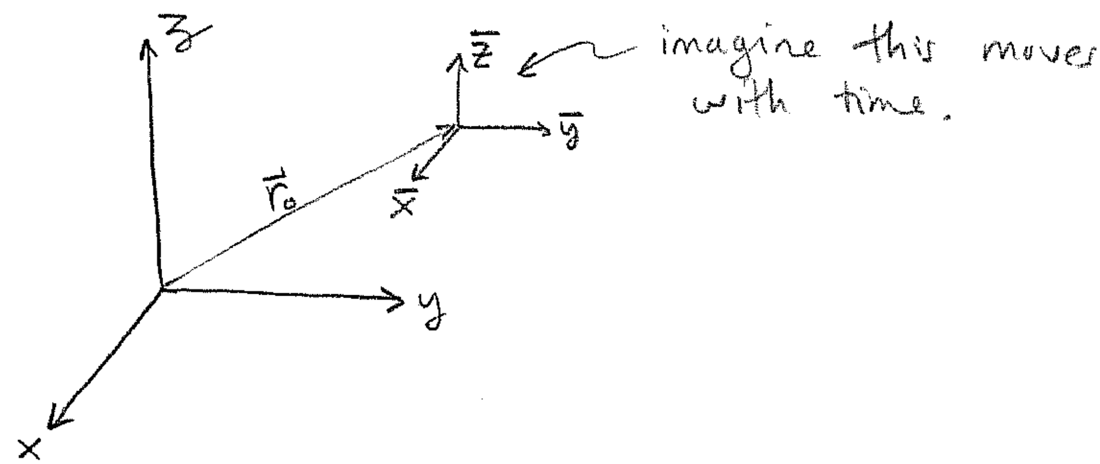
Then we can calculate $\vec{a}_{avg}$ with ease: for $0 \leq t \leq 1.0s$,

$$\begin{aligned}
\vec{a}_{avg} = \frac{\Delta \vec{V}}{\Delta t} &= \frac{1}{1.0s}\left(\vec{V}(1) - \vec{V}(0)\right) \\
&= \frac{1}{1.0s}\left[aR_1(e^{a} - 1)\hat{i} - bR_2(\cos(b+\phi) - \cos\phi)[\hat{j} - \hat{k}]\right]
\end{aligned}$$

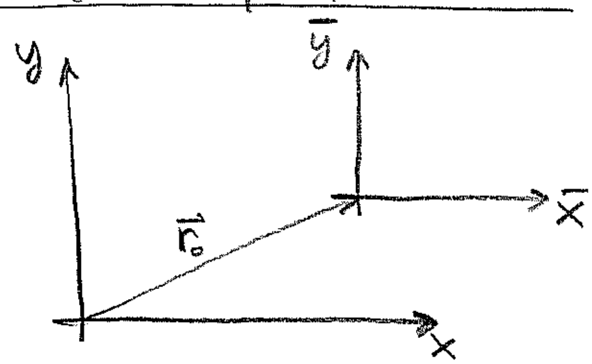
(this example is just to illustrate the math. We'll see physically interesting applications for $\vec{a}_{avg}$ in force chapter)

Relative Velocity and Moving Frames of Rererence

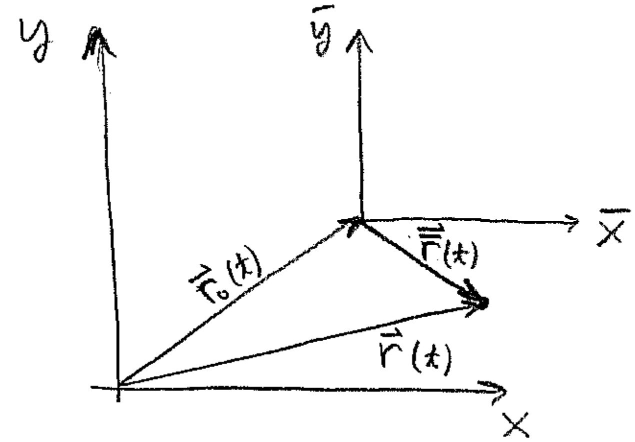
A frame of reference is a coordinate system which labels all of space at a given instant of time. If you think about it, we always have assumed that at least one such frame exists. We continue to make that assumption <u>and</u> we add another frame which might be moving. Let's assume (x, y, z) denote fixed coordinates whereas $(\bar{x}, \bar{y}, \bar{z})$. We can study the motion of a particle with either frame thus it is interesting to compare the velocity $\vec{V}$ relative to (x, y, z) and the velocity $\vec{\bar{V}}$ relative to $(\bar{x}, \bar{y}, \bar{z})$.



<u>Two-dimensional picture easier:</u>



Add a particle to our picture



$$\vec{r}(t) = \vec{r}_o(t) + \vec{\bar{r}}(t)$$

position relative to fixed frame S | position of origin $\bar{S}$ | position of particle relative to $\bar{S}$.

Notations: $\vec{r}(t)$ for $(x, y) \longrightarrow \vec{r}_S(t)$ for S-frame; $\vec{\bar{r}}(t)$ for $(\bar{x}, \bar{y}) \longrightarrow \vec{r}_{\bar{S}}(t)$ for $\bar{S}$-frame

In the $S, \bar{S}$ notation we have

$$\boxed{\vec{r}_S(t) = \vec{r}_o(t) + \vec{r}_{\bar{S}}(t)}$$

Differentiate to find how velocity adds, $\vec{V}_o = \frac{d\vec{r}_o}{dt}$ etc...

$$\boxed{\vec{V}_S(t) = \vec{V}_o + \vec{V}_{\bar{S}}(t)}$$

We'll stop here for now. The formula above is rather interesting. (can derive $\vec{a}_S$ vs. $\vec{a}_{\bar{S}}$ ... Coriolis Effect!.

E7 You throw a ball vertically with speed V_o on train travelling in north-east direction. Find velocity of ball relative the ground.

Let $\bar{S}$ be train frame. Note $\vec{V}_{\bar{S}} = [V_o - gt]\hat{k}$.

Let V_T = speed of train, $\vec{V}_T = \left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)V_T$ thus,

$$\vec{V}_S = \vec{V}_T + \vec{V}_{\bar{S}} = \underline{\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)V_T + (V_o - gt)\hat{k}}.$$

(More to follow)

Relative Velocity Problems:

E8 A river has a speed of 0.550 m/s. Suppose a student swims up the river (against the current) a distance of 1.00 km and then swims back to where he began. If the student can swim 1.10 m/s in still water then how long did his swim in the river take?

Assumption: the student swims at 1.10 m/s relative the frame which is comoving with the river. Let's draw two pictures:

$V_{S1} = 1.10\frac{m}{s} - 0.55\frac{m}{s}$

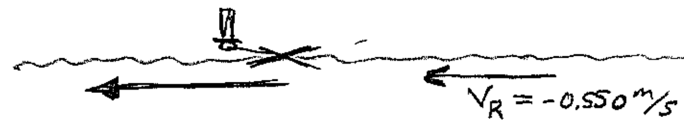
$V_{S1} = 0.55\ m/s$, velocity upstream

Constant velocity assumed thus

$V_{S1} = \frac{\Delta x_1}{\Delta t_1} \rightarrow \Delta t_1 = \frac{\Delta x_1}{V_1}$

$= \frac{1000m}{0.55\ m/s}$

$= 1818.2s$



$V_{S2} = -1.10\frac{m}{s} - 0.55\frac{m}{s}$

$V_{S2} = -1.65\frac{m}{s}$, velocity downstream

Constant velocity assumed thus,

$V_{S2} = \frac{\Delta x_2}{\Delta t_2} \rightarrow \Delta t_2 = \frac{\Delta x_2}{V_{S2}}$

$= \frac{-1000m}{-1.65\ m/s}$

$= 606.1s$

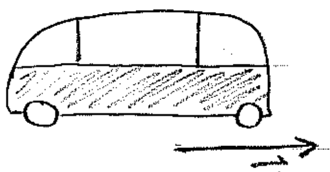
Thus, $\Delta t = \Delta t_1 + \Delta t_2 \cong \boxed{2424s}$

Remark: the problem above is one-dimensional, thus I adopted our usual conventions. Right is positive direction and the sign of the velocity is used to indicate direction.

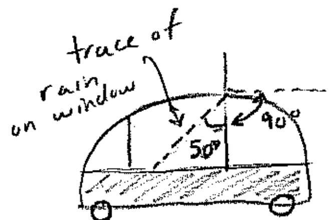
E9 A car travels east at 30.0 km/h. Rain is falling and there is no wind. Suppose that rain drops make lines on the side windows of the car which have angles of 50° with respect to the vertical. How fast is the rain falling?

| | | | | | ← rain falls vertically since there is no wind.

$\vec{V}_{RAIN} = -V_o \hat{j}$ (we're looking for V_o) this is speed of the rain.



$\vec{V}_{CAR} = \left(30.0 \frac{km}{h}\right)\hat{i}$



$\vec{V}_{RAIN,\ CAR\ FRAME} = a\hat{i} + b\hat{j}$ ← in direction of the traces. $\theta = -140°$

Thinking of the CAR FRAME like $\bar{S}$ and ground as S

$$\vec{V}_S = \vec{V}_{CAR} + \vec{V}_{\bar{S}} \Rightarrow \vec{V}_{RAIN} = \vec{V}_{CAR} + \vec{V}_{RAIN,\ CAR}$$

$$\underbrace{-V_o\hat{j} = \left(30.0\frac{km}{h}\right)\hat{i} + a\hat{i} + b\hat{j}}_{\text{gives us two conditions,}}$$

$\hat{j}$: $\quad -V_o = b$

$\hat{i}$: $\quad 0 = 30.0\,km/h + a$

Thus, $a = -30.0\frac{km}{h}$ and $b = -V_o$. We know $\tan(-140°) = \frac{b}{a}$

Hence $\tan(-140°) = \frac{-V_o}{-30.0\,km/h}$

$$V_o = \left(30.0\frac{km}{h}\right)\tan(-140°) = \boxed{25.17\ \frac{km}{h}}$$

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Remark: In mph, we'd have car speed of 18.65 mph. implies rain falling at 15.6 mph.

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Remark: Most cars have slanted side windows and the water trails are not indicative of the free fall of water droplets. The waters path on car windows has more to do with the aerodynamics of the car.

E10 An airline pilot notes a heading of due west by his GPS locater. The plane has an airspeed of 140 km/h and there is a north wind of 27.0 km/h. Find the velocity of the airplane relative to the ground.

Since the plane is flying west relative the ground we can write $\vec{V}_{plane,\,ground} = -V_o\,\hat{i}$. Moreover,

$$\vec{V}_{plane,\,ground} = \vec{V}_{wind} + \underbrace{\vec{V}_{plane,\,wind}}_{\text{velocity of plane relative to frame moving with wind,}}$$

Let $\vec{V}_{plane,\,wind} = a\hat{i} + b\hat{j}$,

$$\underbrace{-V_o\,\hat{i} = \left(27.0\,\tfrac{km}{h}\right)\hat{j} + a\hat{i} + b\hat{j}}$$

we know $\sqrt{a^2+b^2} = 140\,km/h$

$\hat{i}\,|\quad -V_o = a$

$\hat{j}\,|\quad 0 = 27.0\,\frac{km}{h} + b$

$$\therefore\quad V_o^2 + \left(27.0\,\frac{km}{h}\right)^2 = \left(140\,\frac{km}{h}\right)^2$$

$$\Rightarrow\quad V_o = \sqrt{\left(\frac{140\,km}{h}\right)^2 - \left(\frac{27.0\,km}{h}\right)^2} = 142.6\ \frac{km}{h}$$

$$\therefore\quad \boxed{\vec{V}_{plane,\,ground} = -\left(142.6\,\frac{km}{h}\right)\hat{i}}$$

Mathematical Digression on Calculus of Paths

Th^m/ Given path $t \longmapsto \vec{r}(t)$ and scalar function $t \longmapsto f(t)$ we find that

$$\frac{d}{dt}\left[f(t)\,\vec{r}(t)\right] = \frac{df}{dt}\,\vec{r}(t) + f(t)\frac{d\vec{r}}{dt}$$

Consequently we can pull constants out

$$\frac{d}{dt}\left(c\,\vec{r}\right) = c\,\frac{d\vec{r}}{dt} \quad \text{or} \quad \frac{d}{dt}\left(f\,\vec{c}\right) = \frac{df}{dt}\,\vec{c}$$

<u>Proof:</u> Let $\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$ then

$f\vec{r} = fx\,\hat{i} + fy\,\hat{j} + fz\,\hat{k}$ and by definition

$$\frac{d}{dt}(f\vec{r}) = \frac{d}{dt}(fx)\,\hat{i} + \frac{d}{dt}(fy)\,\hat{j} + \frac{d}{dt}(fz)\,\hat{k}$$

product rule thrice!

$$= \left(\frac{df}{dt}x + f\frac{dx}{dt}\right)\hat{i} + \left(\frac{df}{dt}y + f\frac{dy}{dt}\right)\hat{j} + \left(\frac{df}{dt}z + f\frac{dz}{dt}\right)\hat{k}$$

$$= \frac{df}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) + f\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)$$

$$= \underline{\frac{df}{dt}\,\vec{r} + f\frac{d\vec{r}}{dt}}.$$

If $f(t) = c$ then $\frac{df}{dt} = 0 \therefore \frac{d}{dt}(c\vec{r}) = c\frac{d\vec{r}}{dt}$.

If $\vec{r}(t) = \vec{c}$ then $\frac{d\vec{c}}{dt} = 0 \therefore \frac{d}{dt}(f\vec{c}) = \frac{df}{dt}\,\vec{c} = \vec{c}\,\frac{df}{dt}$

<u>Remark:</u> I allow us to write $c\vec{v} = \vec{v}c$ in this course. Either is understood as multiplication of vector by scalar c.

$\boxed{E11}$ Let $\vec{A}, \vec{B}$ be constant vectors then, assuming linearity

$$\frac{d}{dt}\left(t\vec{A} + \vec{B}\right) = \frac{d}{dt}\left(t\vec{A}\right) + \frac{d}{dt}\left(\vec{B}\right) = \frac{dt}{dt}\vec{A} = \vec{A}.$$

My point is that calculus with vectors is not so complicated. Linearity, product rule extend nicely.