LECTURE 7

- we study the kinematics of circular motion. Tangential and normal accelerations are defined and discussed. To begin we provide a little math background.

Th$^{\underline{m}}$/ Let $\vec{A}, \vec{B}$ be vector-valued functions of time with differentiable component functions then

$$\frac{d}{dt}\left(\vec{A}\cdot\vec{B}\right) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

<u>Proof</u>:

$$\frac{d}{dt}\left(\vec{A}\cdot\vec{B}\right) = \frac{d}{dt}\left[\sum_{j=1}^{3} A_j B_j\right] \quad : \text{def}^{\underline{n}} \text{ of dot-product}$$

$$= \sum_{j=1}^{3} \frac{d}{dt}\left(A_j B_j\right) \quad : \text{linearity of } \frac{d}{dt}$$

$$= \sum_{j=1}^{3}\left[\frac{dA_j}{dt} B_j + A_j \frac{dB_j}{dt}\right] \quad : \text{product rule on each } j.$$

$$= \sum_{j=1}^{3} \frac{dA_j}{dt} B_j + \sum_{j=1}^{3} A_j \frac{dB_j}{dt}$$

$$= \underline{\frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}}. //$$

Th$^{\underline{m}}$/ Let $\vec{A}, \vec{B}, \vec{C}$ be vectors and $\alpha, \beta \in \mathbb{R}$ then (1) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$, (2) $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$, $\vec{A}\cdot\alpha\vec{B} = \alpha\vec{A}\cdot\vec{B}$

<u>Proof</u>: Einstein's sneak notation $\vec{A}\cdot\vec{B} = A_iB_i$ omits $\sum_{i=1}^{3}$. I'll demonstrate it here for fun:

(1.) $\vec{A}\cdot\vec{B} = A_iB_i = B_iA_i = \vec{B}\cdot\vec{A}$.

(2.) $\vec{A}\cdot(\vec{B}+\vec{C}) = A_i(\vec{B}+\vec{C})_i = A_i(B_i+C_i) = A_iB_i + A_iC_i = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$.

(3.) $\vec{A}\cdot(\alpha\vec{B}) = A_i(\alpha\vec{B})_i = A_i\alpha B_i = \alpha A_iB_i = \alpha\vec{A}\cdot\vec{B}$.

Einstein's subtle notation allows focus on the essential points of the calculation. The brevity & clarity is especially important to theoretical physics calculations... //

Circular Motion

If $t \longmapsto \vec{r}(t)$ is a path around a circle with center $\vec{C}$ and radius R we have:

$$\| \vec{r}(t) - \vec{C} \| = R \qquad \forall t \in dom(\vec{r})$$

However, recall $\|A\|^2 = \vec{A} \cdot \vec{A}$ thus,

$$(\vec{r} - \vec{c}) \cdot (\vec{r} - \vec{c}) = R^2$$

$$\Rightarrow \vec{r} \cdot \vec{r} - \vec{r} \cdot \vec{c} - \vec{c} \cdot \vec{r} + \vec{c} \cdot \vec{c} = R^2$$

$$\Rightarrow \vec{r} \cdot \vec{r} - 2\vec{r} \cdot \vec{c} + \vec{c} \cdot \vec{c} = R^2$$

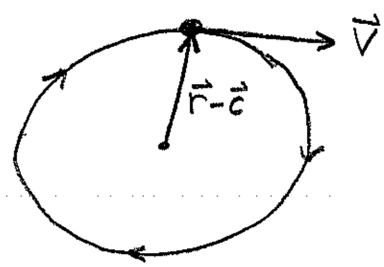
Let's differentiate, note $R^2, \vec{c}$ are <u>constant</u> thus $\Rightarrow$

$$0 = \frac{d\vec{r}}{dt} \cdot \vec{r} + \vec{r} \cdot \frac{d\vec{r}}{dt} - 2\frac{d\vec{r}}{dt} \cdot \vec{c} \qquad \left(\frac{d\vec{c}}{dt} = 0, \ \frac{d}{dt}(R^2) = 0 \right)$$

$$\Rightarrow 0 = 2\frac{d\vec{r}}{dt} \cdot \vec{r} - 2\frac{d\vec{r}}{dt} \cdot \vec{C}$$

$$\Rightarrow 0 = \frac{d\vec{r}}{dt} \cdot (\vec{r} - \vec{c})$$

$$\Rightarrow 0 = \underset{\text{velocity}}{\vec{V}} \cdot \underbrace{(\vec{r} - \vec{c})}_{\text{radial vector}}$$



Differentiate once more,

$$0 = \frac{d\vec{V}}{dt} \cdot (\vec{r} - \vec{c}) + \vec{V} \cdot \left(\frac{d\vec{r}}{dt} \right)$$

$$\Rightarrow \boxed{\vec{a} \cdot (\vec{r} - \vec{c}) = -\vec{V} \cdot \vec{V}} \ - (*)$$

Let $\vec{\mathcal{r}} = \vec{r} - \vec{c}$ and $\hat{\mathcal{r}} = \frac{1}{\mathcal{r}}\vec{\mathcal{r}}$ as usual then (*) reads $\vec{a} \cdot \vec{\mathcal{r}} = -\vec{V} \cdot \vec{V}$. Divide by $\mathcal{r}$ to find,

$$\vec{a} \cdot \hat{\mathcal{r}} = \frac{-\vec{V} \cdot \vec{V}}{\mathcal{r}} = \frac{-V^2}{\mathcal{r}}$$

$$\therefore \boxed{a_{\mathcal{r}} = \frac{-V^2}{\mathcal{r}}}$$

$\Leftarrow$ the center seeking or centripetal acceleration <u>must</u> be proportional to $V^2/\mathcal{r}$ for circular motion.

Circular Motion: Take Two

Let $\vec{r}(t) = R\cos(\theta)\,\hat{i} + R\sin(\theta)\,\hat{j}$. We assume circle centered at origin and we choose coordinates such that $\vec{r}(0) \propto \hat{i}$.

θ generally function of time.

We calculate,

$$\vec{V} = \frac{d\vec{r}}{dt} = -R\sin\theta\,\frac{d\theta}{dt}\,\hat{i} + R\cos\theta\,\frac{d\theta}{dt}\,\hat{j}$$

$$\vec{a} = \frac{d\vec{V}}{dt} = -R\cos\theta\left(\frac{d\theta}{dt}\right)^2\hat{i} - R\sin\theta\,\frac{d^2\theta}{dt^2}\,\hat{i} - R\sin\theta\left(\frac{d\theta}{dt}\right)^2\hat{j} + R\cos\theta\,\frac{d^2\theta}{dt^2}\,\hat{j}$$

$$\Rightarrow \quad \vec{a} = R\left(\frac{d\theta}{dt}\right)^2\underbrace{\left[-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}\right]}_{\substack{\vec{N} \\ \text{unit-normal}}} + R\,\frac{d^2\theta}{dt^2}\underbrace{\left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right]}_{\substack{\vec{T} \\ \text{unit-tangent}}}$$

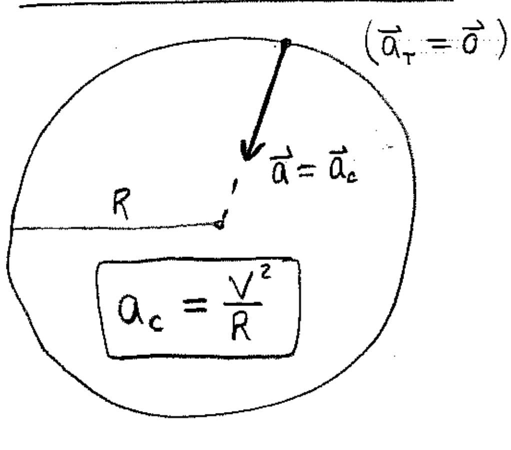
Notice $s = R\theta$ for circle and so $\frac{ds}{dt} = R\frac{d\theta}{dt}$ which tells us that $V = R\frac{d\theta}{dt}$ hence,

$$\vec{a} = \underbrace{\frac{V^2}{R}}_{\substack{\text{centripetal} \\ \text{center-seeking} \\ \text{acceleration}}}\vec{N} + \underbrace{R\frac{d^2\theta}{dt^2}}_{\substack{\text{tangential} \\ \text{acceleration,}}}\vec{T} = a_c\vec{N} + a_T\vec{T}$$

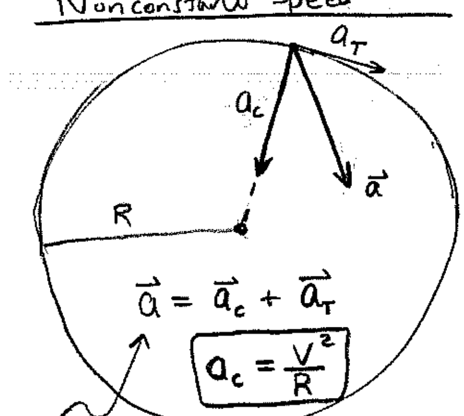
only nontrivial for $\frac{dV}{dt} = R\frac{d^2\theta}{dt^2} \neq 0$.

E1

Constant speed, $\vec{a} = \vec{a}_c\vec{N}$



Nonconstant Speed



resultant of centripetal and normal vector components.

E2 Find the centripetal acceleration at the equator of earth due to the earths rotation. Idealize earth as sphere which spins about axis.

I take a long walk and learn $R_{earth} = 6371\,km$.

We all know that the earth rotates once per day. It follows the speed of the comoving frame at the equator is

$$V = \frac{2\pi R_{EARTH}}{(24)(3600)s} = 0.4633\,\frac{km}{s} = \underbrace{463.3\,\frac{m}{s}}_{1036.6\,mph}$$

Thus you might expect a large a_c

<u>BUT</u> R_{EARTH} is large thus,

$$a_c = \frac{V^2}{R_E} = \frac{(463.3\,\frac{m}{s})^2}{6371\,km} = \underbrace{0.034\,m/s^2}_{a_c \cong 0.0034\,g}$$

quite small in comparison to gravitational acceleration.

E3 a_c for earth relative rotation around the sun is even smaller

$$V_{ORB.} = \frac{2\pi R_{ORBIT}}{(365)(86400\,s)} = \frac{(2\pi)(1.5\times10^{11}\,m)}{(365)(86400)} = 29885.8\,\frac{m}{s}$$

$$a_c = \frac{(29885.8\,m/s)^2}{1.5\times10^{11}\,m} = \underline{0.00595\,m/s^2}.$$