Mathematics of Supersymmetric Field Theory

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Abstract

Supersymmetry (SUSY) is an exciting new symmetry of theoretical physics that has yet to be verified experimentally. It arises naturally in superstring theory and is the only known physical extension of the Poincare algebra. From a mathematician's perspective many equations studied by physicists involving SUSY are in a strict sense ambiguous. Our objective is to remove this ambiguity and give the physicists calculations a mathematical home.

A Short Course in the Physics of Supersymmetry

- fermions have half-integer spin; $s = 1/2, 3/2, ...$
- bosons have integer spin; $s = 0, 1, 2, ...$

Supersymmetry states that nature has a balance between fermions and bosons. In a system with unbroken SUSY there would be an equal number of bosonic and fermionic degrees of freedom. The Minimal Supersymmetric Standard Model (MSSM) is an example of a physical model based on SUSY. If the MSSM is correct then nature has at least twice as many particles as the current Standard Model includes. There is hope that the Large Hadron Collider at CERN will observe the first low-energy(TeV) signatures of SUSY soon.

Known Particle	Spin	SUSY	predicted by MSSM	Spin
electron	1/2	$\leftrightarrow$	selectron	0
photon	1	$\leftrightarrow$	photino	1/2
quark	1/2	$\leftrightarrow$	squark	0
gluon	1	$\leftrightarrow$	gluino	1/2
Higgs	0	$\leftrightarrow$	Higgsino	1/2

SUSY is a symmetry which rotates known particle states into new states which are not yet observed. Observed symmetries in particle physics such as $SU(3)$-flavor or the gauge symmetries $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ rotate particle states of a given spin into the same spin. Observe that SUSY is quite different, it connects particles of spin differing by 1/2. This means that representations of SUSY will necessarily contain several irreps of the Poincare group. Salam and Strathdee first introduced the superfield which represents $N = 1$ SUSY,

$$U = f + \theta\phi + \bar{\theta}\bar{\mathcal{X}} + \theta\theta m + \bar{\theta}\bar{\theta}n + \theta\sigma^n\bar{\theta}v_n + \theta\theta\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}\theta\psi + \theta\theta\bar{\theta}\bar{\theta}d \tag{1}$$

This is called the *component field expansion*. You can see how a single superfield U contains all the following relativistic fields, just think of the θ's as place-holders for now,

scalar fields	f, m, n, d	spin 0	commuting fields
Weyl spinors	$\phi, \bar{\mathcal{X}}, \bar{\lambda}, \psi$	spin 1/2	anticommuting fields
vector field	v_n	spin 1	commuting field

Models built with superfields are almost automatically supersymmetric. For example, the gauged Wess Zumino model has the following superfield Lagrangian

$$\mathcal{L} = \frac{1}{16kg^2}\Big[tr(WW|_{\theta\theta}) + tr(\overline{WW}|_{\bar{\theta}\bar{\theta}})\Big] + \bar{\Phi}e^V\Phi|_{\theta\theta\bar{\theta}\bar{\theta}}$$

where W is a spinor superfield which has component fields λ, F_{mn}, D which could represent the photino, photon and an auxillary field respectively, and Φ is a chiral superfield which has component fields ψ, A, F which could represent the electron, selectron and an auxillary field respectively. The same Lagrangian written explicitly in component field form is,

$$\mathcal{L} = \frac{1}{16kg^2}tr\big(-4i\bar{\lambda}\bar{\sigma}^m\mathcal{D}_m\lambda - F^{mn}F_{mn} + 2D^2\big)$$

$$+\bar{F}F - i\bar{\psi}\bar{\sigma}^m\mathcal{D}_m\psi - \mathcal{D}_m\bar{A}\mathcal{D}^mA + \frac{1}{2}\bar{A}DA + \frac{i}{\sqrt{2}}(\bar{A}\lambda\psi - \bar{\psi}\bar{\lambda}A)$$

Supernumbers

- In a nutshell our research goal is to unravel the math of superfields. To do this we begin with supernumbers.
- Grassmann generators ζ^i anticommute $\zeta^i\zeta^j = -\zeta^j\zeta^i$. Note $(\zeta)^2 = 0$ even if $\zeta \neq 0$.
- Supernumbers are constructed as follows,

$$z \;=\; \underbrace{\sum_{p=0}^{\infty}\sum_{i_1<\cdots<i_{2p}} z_{i_1\ldots i_{2p}}\zeta^{i_1}\zeta^{i_2}\cdots\zeta^{i_{2p}}}_{in\ ^0\Lambda\ (commuting\ supernumbers)} + \underbrace{\sum_{p=0}^{\infty}\sum_{i_1<\cdots<i_{2p+1}} z_{i_1\ldots i_{2p+1}}\zeta^{i_1}\zeta^{i_2}\cdots\zeta^{i_{2p+1}}}_{in\ ^1\Lambda\ (anticommuting\ supernumbers)}$$

- Conjugation of supernumbers satisfies $(zw)^* = w^*z^*$, a supernumber is said to be real if $z^* = z$.

Superspace and Supersmooth Functions

- Flat superspace is generically $\mathbb{K}^{p|q} = (^0\Lambda)^p \times (^1\Lambda)^q$ then $z \in \mathbb{K}^{p|q}$ has $z = (x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ where $x^m \in {}^0\Lambda$ for $m = 1, 2, \ldots, p$ and $\theta^\alpha \in {}^1\Lambda$ for $\alpha = 1, 2, \ldots, q$. The dimension of $\mathbb{K}^{p|q}$ is $(p|q)$, however as a Banach space $\mathbb{K}^{p|q}$ is infinite dimensional.

- Let $U \subseteq \mathbb{K}^{p|q}$, f is said to be G^1 on U if there exist p+q functions $G_Mf : U \to \Lambda$, $M = 1, 2, \ldots, p+q$

$$f(a+h, b+k) = f(a,b) + \sum_{m=1}^{p} h^m(G_mf)(a,b) + \sum_{\alpha=1}^{q} k^\alpha(G_{p+\alpha}f)(a,b) + ||(h,k)||\eta(h,k)$$

where $||\eta(h,k)|| \to 0$ as $||(h,k)|| \to 0$. Here G^∞ or *supersmooth* functions are defined inductively just as in the C^∞ case. Alice Rogers invented this definition to mirror the standard calculus over $\mathbb{R}$. In contrast to most of the literature her definition allows us to view derivatives of Grassmann variables in terms of a limiting process.

- The superspace $\mathbb{R}^{4|4}$ is known as $N = 1$ rigid superspace. A typical point in $\mathbb{R}^{4|4}$ is $z = (x^0, x^1, x^2, x^3, \theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2) \in \mathbb{R}^{4|4}$ where by defintion the coordinates x^m are *real* even Grassmann variables while the $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ are *conjugate* odd Grassmann variables,

$$(x^m)^* = x^m \quad m = 0, 1, 2, 3 \qquad\qquad (\bar{\theta}^1)^* = \theta^1 \quad (\bar{\theta}^2)^* = \theta^2$$

Suppose $U : \mathbb{R}^{4|4} \mapsto V$ then since $\theta^\alpha\theta^\alpha = 0$ and $\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = 0$ it follows that the only nontrivial terms are precisely those we wrote in the *component field expansion* (see $eq^n(1)$). The *component fields* are functions of (x^0, x^1, x^2, x^3). Physicists call the component functions "x-space" dependent and intuitively identify (x^0, x^1, x^2, x^3) with Minkowski space.

Supermanifolds: Curved Superspace

- $\mathcal{M}$ is a $(p|q)$ dimensional supermanifold if it is locally $\mathbb{K}^{p|q}$. Just as in the standard manifold theory we require that $\mathcal{M}$ be covered by an atlas of compatible charts. What is new is that *compatible* means that the charts have supersmooth transition functions.

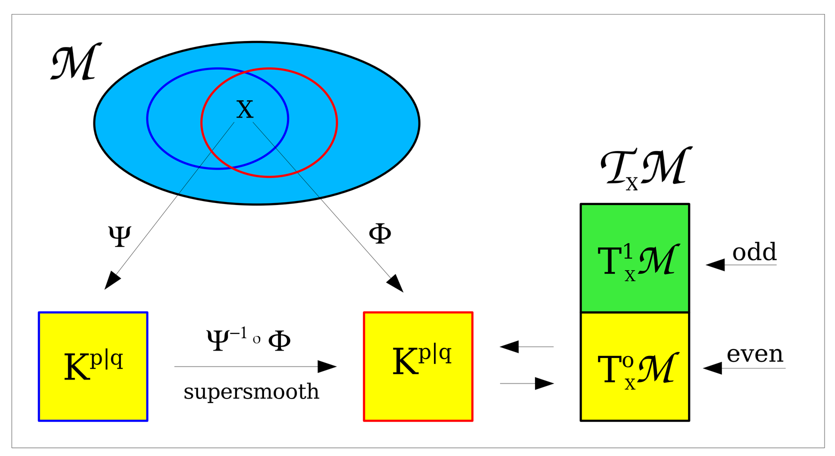

Figure 1: Parameter space $\mathbb{K}^{p|q}$ and tangent space $T_x\mathcal{M}$ for a supermanifold $\mathcal{M}$ contrasted.

- $G^\infty(\mathcal{M})$ denotes supersmooth functions from $\mathcal{M}$ to Λ. An *even* function outputs even supernumbers. An *odd* function outputs odd supernumbers.

- *Tangent vectors* are derivations on $G^\infty(\mathcal{M})$. *Even* tangent vectors map even functions to even functions and odd to odd. *Odd* tangent vectors map even functions to odd functions and vice-versa. The set of all tangents at $x \in \mathcal{M}$ is the tangent space $T_x\mathcal{M}$.

- $\mathcal{M}$ is locally $\mathbb{K}^{p|q}$ however the tangent space $T_x\mathcal{M} = \Lambda^{p+q}$. This is in marked contrast to ordinary manifold theory where it is customary to identify the tangent space and parameter space. We have found identification of the even tangent space $T_x^0\mathcal{M}$ with $\mathbb{K}^{p|q}$ useful.

Super Lie Groups

- A supermanifold $\mathcal{G}$ which is also an abstract group is called a super Lie group if the group operations are G^∞ with respect to the supermanifold structure on $\mathcal{G}$.

- Since G^∞ functions are always class C^∞ functions, it follows that the Banach manifold $\mathcal{BG}$ corresponding to a super Lie group $\mathcal{G}$ is necessarily a Banach Lie group.

- Let $\mathcal{G}$ be a super Lie group and $\mathfrak{g}$ its tangent space $T_e\mathcal{G}$ at the identity e of $\mathcal{G}$. Let $\phi_v : \mathbb{R} \times \mathcal{BG} \to \mathcal{BG}$ denote the flow of the vector field X^v on $\mathcal{BG}$. We define *exp* as the mapping from $\mathfrak{g}^0$ into $\mathcal{BG}$ defined by $exp(v) \equiv \phi_v(1, e)$. *In constrast physicists typically define the exp in terms of a local formula, they assume the Baker-Campbell-Hausdorff relation whereas we derive it.*

New Theorems in the G^∞-category

- In the case of finitely generated supernumbers the theorems below are proved using finite dimensional Lie group theory. However, in the G^∞ case our supermanifolds are infinite dimensional manifolds so the known proofs were not appropriate for the G^∞-case.

Theorem: $exp : \mathfrak{g}^0 \to \mathcal{G}$ is a class G^∞ mapping.

- This theorem provides a geometrical foundation to the formal exponential found in the physics literature. In particular, we can use it to prove that the super Poincare group is a G^∞-supermanifold.

Theorem: If $\mathcal{G}$ is a $(p|q)$ super Lie group and $\mathcal{S}$ is a closed $(r|s)$ sub-super Lie group of $\mathcal{G}$ then $\mathcal{G}/\mathcal{S}$ is a $(p-r|q-s)$ supermanifold. Moreover $\mathcal{G} \to \mathcal{G}/\mathcal{S}$ is a G^∞-mapping and is a principal fiber bundle with structure group the super Lie group $\mathcal{S}$. All local trivializing maps are G^∞-maps.

- As application of this theorem take the quotient of the super Poincare group by the super Lorentz group, the resulting space is identified with $\mathbb{R}^{4|4}$. Moreover, this construction induces the action of the Lorentz group on the component fields of a $N = 1$ superfield. That is it makes θ^α and $\bar{\theta}^{\dot{\alpha}}$ into Weyl spinors. What is new is that we have obtained these results in the G^∞ category. That is we have provided a solid mathematical foundation for the superfield construction first introduced by of Salam and Strathdee.

Principle Fiber Bundles and Gauge Theory

- The central idea of modern physics is that symmetry guides and limits the physically possible states. Local or gauge symmetries are used to derive the known forces in current physical theory. Traditionally physicists begin with a global symmetry then modify it to be a local symmetry by changing certain constants to functions. Intuitively this amounts to attaching a distinct copy of the symmetry group to each point in space. Curiously this leads to the introduction of a new object that transforms inhomogeneously under the local symmetry group,

$$A_\mu \mapsto gA_\mu g^{-1} - (\partial_\mu g)g^{-1}. \tag{2}$$

What is truly beautiful is that one finds that this object is simply the well-known vector potential of electromagnetism. Weyl first derived $E\&M$ from this gauge-principle in 1929, but it was regarded a novelity for some decades. Then in 1956-1957 the work of Yang, Mills and Utiyama showed Weyl's gauge principle could be generalized to other nonabelian gauge groups. It turns out that the Principle Fiber Bundle (invented by mathematicians unaware of gauge theory) yields the same mathematics.

- A Principle Fiber Bundle (PFB) has (P, M, π, G, ψ_i) where P is the bundle space, M is the base space which we identify with spacetime in Yang-Mills theory, π is the projection, G is the gauge group which acts on the *fibers* of P; $\pi(xg) = \pi(x)$ for all $g \in G$, and finally ψ_i are local trivializing maps from $\pi^{-1}(U_i) \mapsto U_i \times G$.

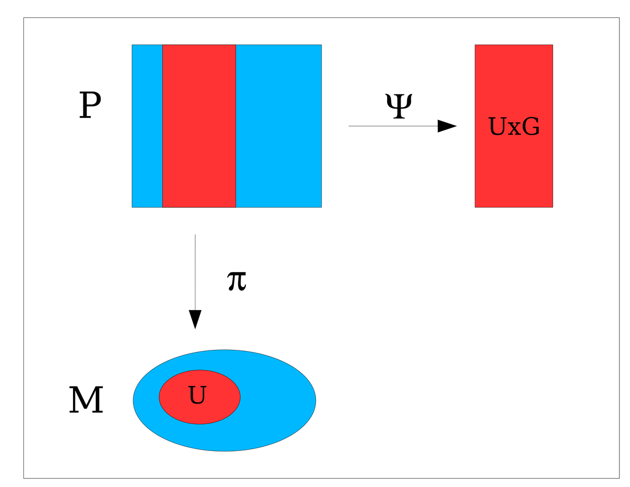

Figure 2: Principle Fiber Bundle is locally the Cartesian product $U \times G$.

- A connection one-form ω on a PFB pulls back under a local section to A on the base manifold. If we change sections then we obtain precisely $eq^n(2)$, the same transformation that physicists postulated in gauge theory.

Future Work: Super Fiber Bundles and Super Yang Mills Theory

- We would like to develop a geometric picture of super Yang-Mills theory in terms of G^∞-supermanifold theory. At the present time there does not appear to be a geometric account mirroring the standard gauge theory over PFB's. There have been attempts but they involve physics that would be distracting to mathematicians. It is our hope we can give a description which is more accessible to geometers.

- The basic idea is to replace spacetime with superspace. This builds SUSY into our model. We expect that many of our super Lie group constructions will find another application here. Other applications are supergravity (gauge theory of local SUSY) and string theory.