

# QUARKS AND $su(3)$

- A FLAVORFUL APPLICATION OF REPRESENTATION THEORY

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## Sources:

- INTRO. TO ELEMENTARY PARTICLES, GRIFFITHS
- QUANTUM MECHANICS: SYMMETRIES 2<sup>nd</sup> ED., GREINER & MÜLLER
- GROUP THEORY AND PHYSICS, S. STERNBERG
- REPRESENTATION THEORY: A FIRST COURSE, FULTON & HARRIS

(given in order of increasing mathematical sophistication and decreasing physical interpretation, mostly follows GREINER )

## QUANTUM MECHANICS

The wavefunction  $\psi$  is a vector valued function which has dynamics governed by the eq<sup>n</sup> below,

$$i \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

A symmetry of the Hamiltonian  $\hat{H}$  is a transformation  $\hat{U}$  which transforms  $\psi \mapsto \psi' = \hat{U} \psi$  such that  $\psi'$  is also a sol<sup>n</sup> to the equation of motion above. Also,

- I.)  $\hat{U}^\dagger \hat{U} = 1$
- II.)  $\hat{U}$  is a global transformation ( $\alpha \neq \alpha(\vec{x})$ )
- III.)  $\hat{U} = \exp(-i \sum_r \alpha_r \hat{L}_r)$  for some  
 $\hat{L}_r$  = parameters of group  
 $\hat{L}_r$  = generators of group

Prop.:  $\hat{U}$  is a symmetry of  $\hat{H} \Leftrightarrow [\hat{H}, \hat{U}] = 0$

$$\begin{aligned}
 \text{Pf: } i \frac{\partial \psi'}{\partial t} &= \hat{H} \psi' \Leftrightarrow i \frac{\partial}{\partial t} (\hat{U} \psi) = \hat{H} \hat{U} \psi \\
 &\Leftrightarrow i \hat{U} \frac{\partial \psi}{\partial t} = \hat{H} \hat{U} \psi \\
 &\Leftrightarrow \hat{U} \hat{H} \psi = \hat{H} \hat{U} \psi \\
 &\Leftrightarrow [\hat{H}, \hat{U}] = 0 //
 \end{aligned}$$

The set of all symmetries  $G$  is a Lie Group which has Lie Algebra  $\mathfrak{g}$  formed by the generators of the group. Other important ideas for us

- $\text{rank}(G) = \max \# \text{ of commuting } \hat{L}_\mu \text{ (generators)}$
- $G \text{ (semi)simple} \iff \mathfrak{g} \text{ (semi)simple}$
- **multiplets** are  $G$ -invariant irreducible subspaces, a multiplet of states transforms into itself under  $G$ .

**Th<sup>m</sup>(RACAH)** For any semisimple Lie group of rank  $l$   $\exists l$ -Casimir operators  $\hat{C}_\lambda$ ,  $\lambda=1, 2, \dots, l$  which commute with the whole group. The multiplets of  $G$  can be labeled by the eigenvalues  $C_1, C_2, \dots, C_l$  because the Casimirs are constant within each multiplet.

Remark: the multiplets are energetically degenerate.

Take  $\Psi_0$  in some multiplet, suppose it has energy  $E_0$  meaning  $\hat{H}\Psi_0 = E_0\Psi_0$ . Consider, if  $\hat{U}$  is a symmetry then  $[\hat{H}, \hat{U}] = 0$ . Notice,  $\Psi' = \hat{U}\Psi_0$  has

$$\begin{aligned}\hat{H}\Psi' &= \hat{H}\hat{U}\Psi_0 \\ &= \hat{U}\hat{H}\Psi_0 \\ &= \hat{U}E_0\Psi_0 = E_0(\hat{U}\Psi_0) = E_0\Psi' \quad \therefore E' = E_0\end{aligned}$$

Therefore, each state in a multiplet has same energy.

## ISO SPIN SYMMETRY: $P \leftrightarrow N$

Protons ( $P$ ) and neutrons ( $N$ ) have nearly the same mass thus we can place them in a multiplet of a symmetry called isospin. (similar to spin)

$$\begin{aligned}\hat{T}_+ N &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv P \\ \hat{T}_- P &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv N\end{aligned}$$

The operators  $\hat{T}_{\pm}$  are related to the Pauli-matrices,

$$\left. \begin{aligned}\hat{T}_1 &= \hat{T}_+ + \hat{T}_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \hat{T}_2 &= i(\hat{T}_+ - \hat{T}_-) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \hat{T}_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned} \right\} \begin{array}{l} \text{su(2) Lie Algebra} \\ [\hat{T}_i, \hat{T}_j] = 2i\epsilon_{ijk}\hat{T}_k \end{array}$$

Customarily we rescale these to  $T_K \equiv \frac{1}{2} T_K$  and technically I should say these give the isospin,

$$\begin{aligned}\hat{T}_3 P &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} P \\ \hat{T}_3 N &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} N\end{aligned}$$

States in a rep. of  $su(2)$  can be labeled by the eigenvalues of  $\hat{T}^2$  and  $\hat{T}_3$  which are  $T(T+1)$  and  $T_3$ ,

$$N = | \frac{1}{2} \ - \frac{1}{2} \rangle \text{ isospin } -\frac{1}{2}$$

$$P = | \frac{1}{2} \ \frac{1}{2} \rangle \text{ isospin } \frac{1}{2}$$

- $\hat{T}^2 = T_1^2 + T_2^2 + T_3^2$  is the Casimir operator for  $su(2)$ , the eigenvalues  $T(T+1)$  specify the multiplets. for example  $\{| \frac{1}{2} T_3 \rangle\} = \{N, P\}$ , the "iso-doublet"

PIONS  $\pi^+$ ,  $\pi^0$ ,  $\pi^-$  form an iso-triplet  $|1 T_3\rangle$

The pions also have similar mass and typically are found in certain nuclear reactions,

		ISOSPIN CONSERVED!
$\pi^+ + P$	$\longrightarrow \pi^+ + P$	$1 + \frac{1}{2} = 1 + \frac{1}{2}$
$\pi^+ + N$	$\begin{array}{l} \longrightarrow \pi^+ + N \\ \searrow \\ \pi^0 + P \end{array}$	$1 - \frac{1}{2} = 1 - \frac{1}{2}$ $= 0 + \frac{1}{2}$
$\pi^0 + P$	$\begin{array}{l} \longrightarrow \pi^0 + P \\ \searrow \\ \pi^+ + N \end{array}$	$0 + \frac{1}{2} = 0 + \frac{1}{2}$ $= 1 - \frac{1}{2}$
$\pi^0 + N$	$\begin{array}{l} \longrightarrow \pi^0 + N \\ \searrow \\ \pi^- + P \end{array}$	$0 - \frac{1}{2} = 0 - \frac{1}{2}$ $= -1 + \frac{1}{2}$
$\pi^- + P$	$\begin{array}{l} \longrightarrow \pi^- + P \\ \searrow \\ \pi^0 + N \end{array}$	$-1 + \frac{1}{2} = -1 + \frac{1}{2}$ $= 0 - \frac{1}{2}$
$\pi^- + N$	$\longrightarrow \pi^- + N$	$-1 - \frac{1}{2} = -1 - \frac{1}{2}$

In each of the reactions above we find that isospin ( $T_3$ ) is conserved if we assign the pions to an iso-triplet,

$\hat{T}_3  \pi^+\rangle =  \pi^+\rangle$	$\pi^+ =  1+\rangle$
$\hat{T}_3  \pi^0\rangle = 0$	$\pi^0 =  10\rangle$
$\hat{T}_3  \pi^-\rangle =  \pi^-\rangle$	$\pi^- =  1-\rangle$

## HYPERCHARGE = $Y$

If we examine the isodoublet  $\{N, P\}$  and the isotriplet  $\{\pi^+, \pi^0, \pi^-\}$  we notice that there is a certain regularity in the electric charge  $Q$  of the states. We can describe the pattern by

$$\hat{Q} \equiv \frac{1}{2} \hat{Y} + \hat{T}_3$$

Gell-Mann

Nishijima relation ( $e=1$ )  
(defines  $Y$  for us,  $Y = B + S$  also)

for each  $SU(2)$ -isospin multiplet we can assign some value for hypercharge.

$\{N, P\}$  has  $Y = 1$

$$\hat{Q}N = \left(\frac{1}{2}\hat{Y} + \hat{T}_3\right)N = \left(\frac{1}{2} - \frac{1}{2}\right)N = 0 \quad \checkmark$$

$$\hat{Q}P = \left(\frac{1}{2}\hat{Y} + \hat{T}_3\right)P = \left(\frac{1}{2} + \frac{1}{2}\right)P = P \quad \checkmark$$

$\{\pi^+, \pi^0, \pi^-\}$  has  $Y = 0$

$$\hat{Q}\pi^+ = \left(\frac{1}{2}\hat{Y} + \hat{T}_3\right)\pi^+ = (0+1)\pi^+ = \pi^+ \quad \checkmark$$

$$\hat{Q}\pi^0 = \left(\frac{1}{2}\hat{Y} + \hat{T}_3\right)\pi^0 = (0+0)\pi^0 = 0 \quad \checkmark$$

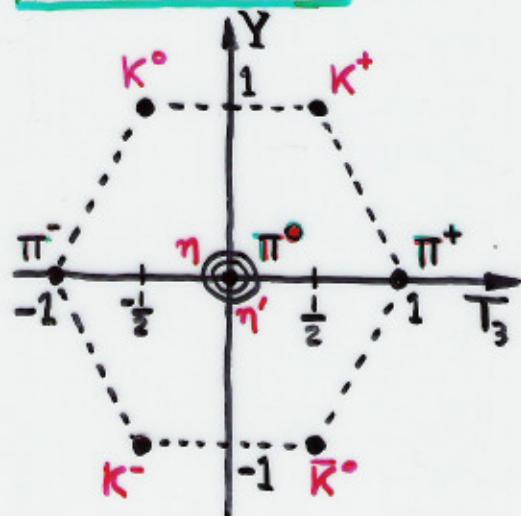
$$\hat{Q}\pi^- = \left(\frac{1}{2}\hat{Y} + \hat{T}_3\right)\pi^- = (0-1)\pi^- = -\pi^- \quad \checkmark$$

- We find that the particles are eigenstates of the charge operator  $\hat{Q}$  with eigenvalues  $Q$ , the charges of the particles. Indeed the Gell-Mann Nishijima relation reproduces the known charges of  $N, P$  and  $\pi^+, \pi^0, \pi^-$ .

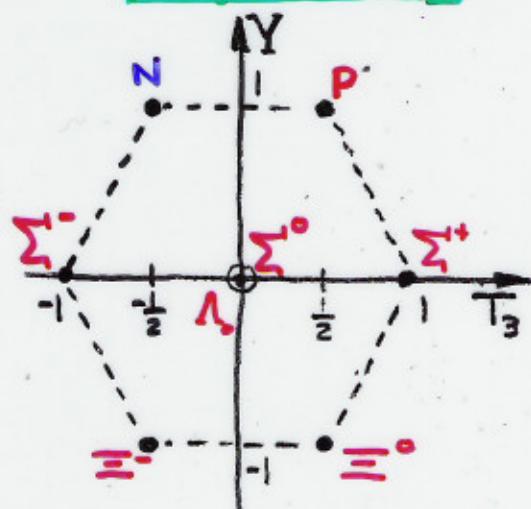
## THE PARTICLE ZOO

Protons, neutrons, pions, and...

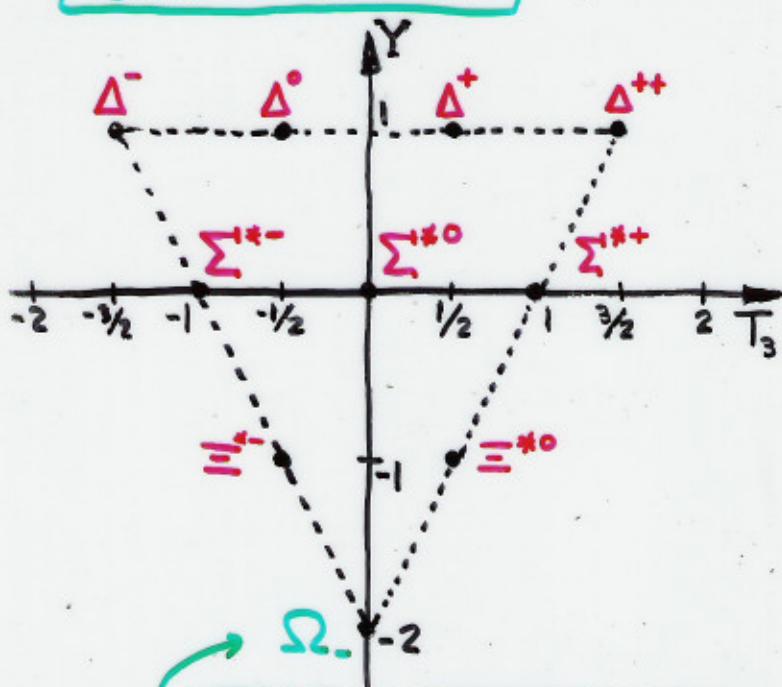
### Meson Nonet



### Baryon Octet



### BARYON DECUPLET



Gell-Mann predicted in 1961  
was found in 1964.

- The particles in each pattern share similar spin, mass and parity
- This classification is known as the Eightfold-Way. It was proposed by Gell-Mann in 1961.
- hmm,... if only we could explain these patterns by some symmetry...

SU(3) has the Lie Algebra  $\text{su}(3) = \text{span}\{\lambda_i\}_{i=1}^8$

The matrices below are the Gell-Mann matrices

$$\begin{array}{lll} \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} & \end{array}$$

- These Hermitian, traceless matrices span all such matrices.
- Clearly the Pauli-matrices of  $\text{su}(2)$  are embedded above
- Again it is customary to rescale these to form the "F-Spin"

$$F_i = \frac{1}{2} \lambda_i \quad i=1, 2, \dots, 8$$

A very useful change of basis is,

$$\begin{aligned} T_z &= F_1 \pm i F_2 \\ T_3 &= F_3 \\ V_z &= F_4 \pm i F_5 \\ U_z &= F_6 \pm i F_7 \\ Y &= \frac{2}{\sqrt{3}} F_8 \end{aligned}$$

"spherical basis  
of  $\text{su}(3)$ "

- We can see 3 copies of  $\text{su}(2)$  which are referred to as  $T-U-V$  spin

Lie algebra of  $su(3)$  in the spherical basis

$$(1.) [T_3, T_{\pm}] = \pm T_{\pm}$$

$$[T_+, T_-] = 2T_3$$

$$(2.) [T_3, U_{\pm}] = \mp \frac{1}{2} U_{\pm}$$

$$[U_+, U_-] = \frac{3}{2} Y - T_3 \equiv 2U_3$$

$$(3.) [T_3, V_{\pm}] = \pm \frac{1}{2} V_{\pm}$$

$$[V_+, V_-] = \frac{3}{2} Y + T_3 \equiv 2V_3$$

$$(4) [Y, T_{\pm}] = 0$$

$$(5) [Y, U_{\pm}] = \pm U_{\pm}$$

$$(6) [Y, V_{\pm}] = \pm V_{\pm}$$

$$[T_+, V_+] = [T_+, U_-] = [U_+, V_+] = 0$$

$$[T_+, V_-] = -U_-$$

$$[T_+, U_+] = V_+$$

$$[U_+, V_-] = T_-$$

$[T_3, Y] = 0 \Rightarrow$  can use common eigenstates  
to study irred. rep of  $su(3)$   
namely  $|T_3, Y\rangle$

$$\hat{T}_3 |\bar{T}_3 Y\rangle = T_3 |\bar{T}_3 Y\rangle$$

$$\hat{Y} |\bar{T}_3 Y\rangle = Y |\bar{T}_3 Y\rangle$$

## SHIFTING THE STATE $|T_3 Y\rangle$

Let's study how the  $T, U, V$  subalgebras act on  $|T_3 Y\rangle$

$$\begin{aligned}\hat{T}_3 \hat{T}_{\pm} |T_3 Y\rangle &= (\hat{T}_{\pm} \hat{T}_3 \pm \hat{T}_{\pm}) |T_3 Y\rangle : \text{using (1.)} \\ &= (\hat{T}_{\pm} T_3 \pm \hat{T}_z) |T_3 Y\rangle \\ &= (T_3 \pm 1) \hat{T}_{\pm} |T_3 Y\rangle \quad \because \hat{T}_{\pm} |T_3 Y\rangle \text{ has } T_3' = T_3 \pm 1\end{aligned}$$

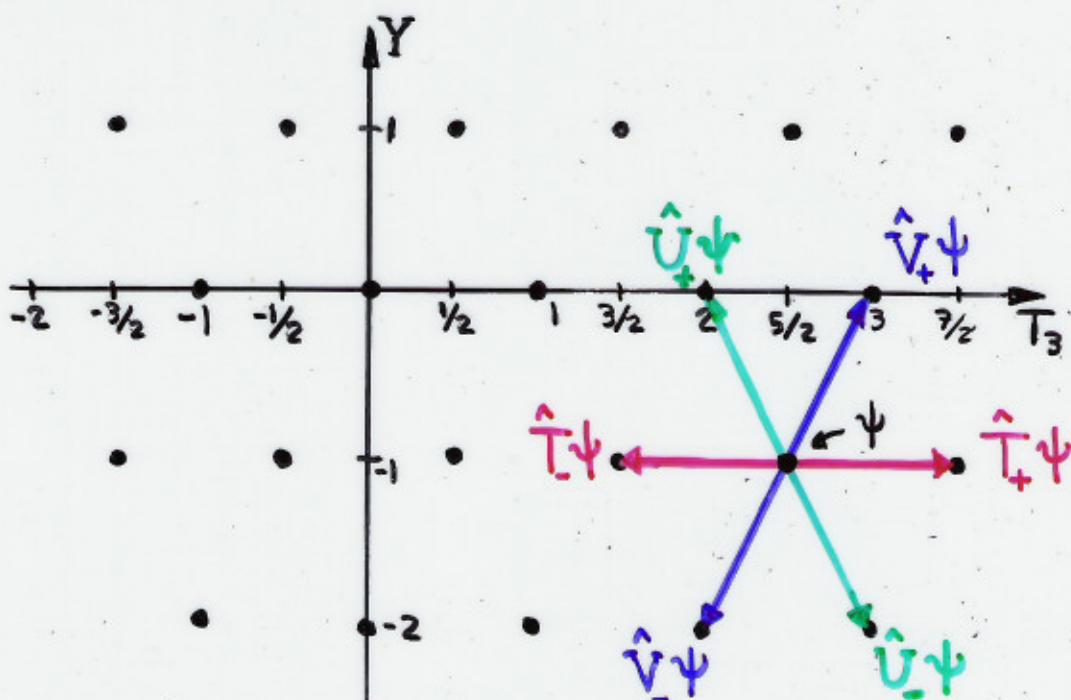
Similar calculations based on (2.), (3.), (4.), (5.), (6.) reveal,

$$\hat{T}_{\pm} |T_3 Y\rangle \text{ has } T_3' = T_3 \pm 1 \text{ AND } Y' = Y$$

$$\hat{U}_{\pm} |T_3 Y\rangle \text{ has } T_3' = T_3 \mp \frac{1}{2} \text{ AND } Y' = Y \pm 1$$

$$\hat{V}_{\pm} |T_3 Y\rangle \text{ has } T_3' = T_3 \pm \frac{1}{2} \text{ AND } Y' = Y \pm 1$$

We can present this result graphically in the  $(T_3 Y)$ -plane,



Remark: we've rescaled the  $Y$ -axis to make the diagram symmetric.

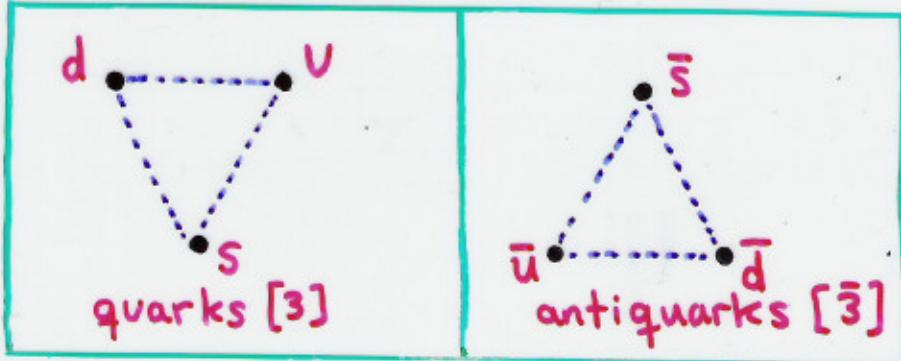
## MULTIPLETS OF $SU(3)$

- 1.)  $SU(3)$  is rank 2, so the eigenvalues of  $\hat{T}_3$  and  $\hat{Y}$  suffice to classify the possible irred. reps.
- 2.)  $SU(3)$  written in the spherical basis clearly has three copies of the isospin algebra  $SU(2)$  which we've denoted  $T-U-V$
- 3.)  $SU(3)$  mixes the isospin subalgebras, thus each  $SU(3)$  irred. rep. must contain  $T-U-V$  irred. rep. In other words the multiplets of  $SU(3)$  displayed in the  $(T_3, Y)$ -plane must be invariant under  $120^\circ$  rotations

Conclusion:  $SU(3)$  multiplets must be triangles or hexagons in the  $(T_3, Y)$ -plane.

## QUARKS: THE FUNDAMENTAL REP. OF SU(3)

$\text{su}(3)$  multiplets can be formed by tensoring together  $[3]$  and  $[\bar{3}]$  in various combinations, (see next page)



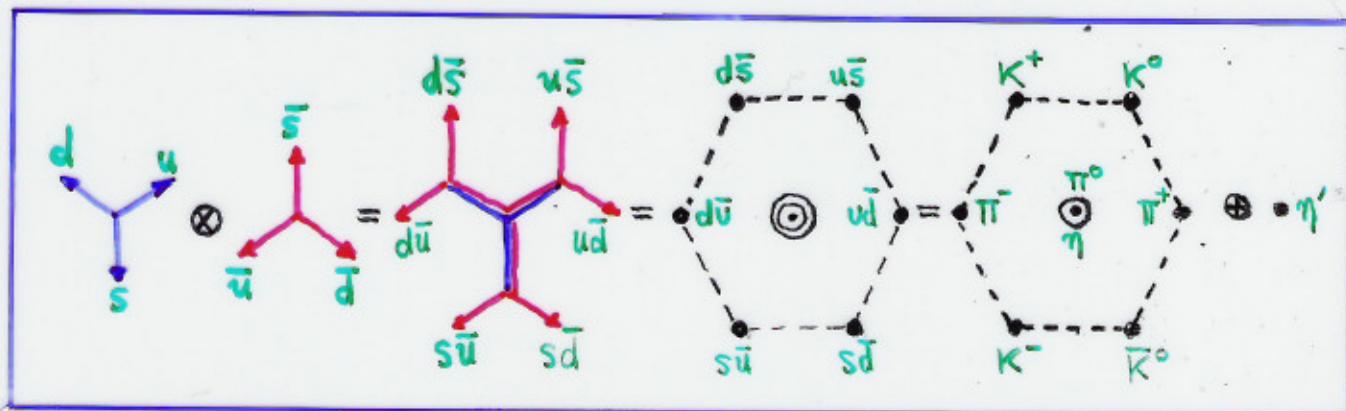
these are the smallest nontrivial reps. We now list the defining traits of the "quarks"

name	$T_3$	$Y$	$Q = \frac{1}{2}Y + T_3$	$B$	$S$
$u = \text{up}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0
$d = \text{down}$	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	0
$s = \text{strange}$	0	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	-1
$\bar{u} = \text{anti-up}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	0
$\bar{d} = \text{anti-down}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0
$\bar{s} = \text{anti-strange}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	+1

"Baryon #"  
"Strangeness"  
notice:  $Y = B + S$

## BUILDING MESONS WITH QUARKS

RECIPE FOR  $\pi$ : ONE QUARK & ONE ANTIQUARK

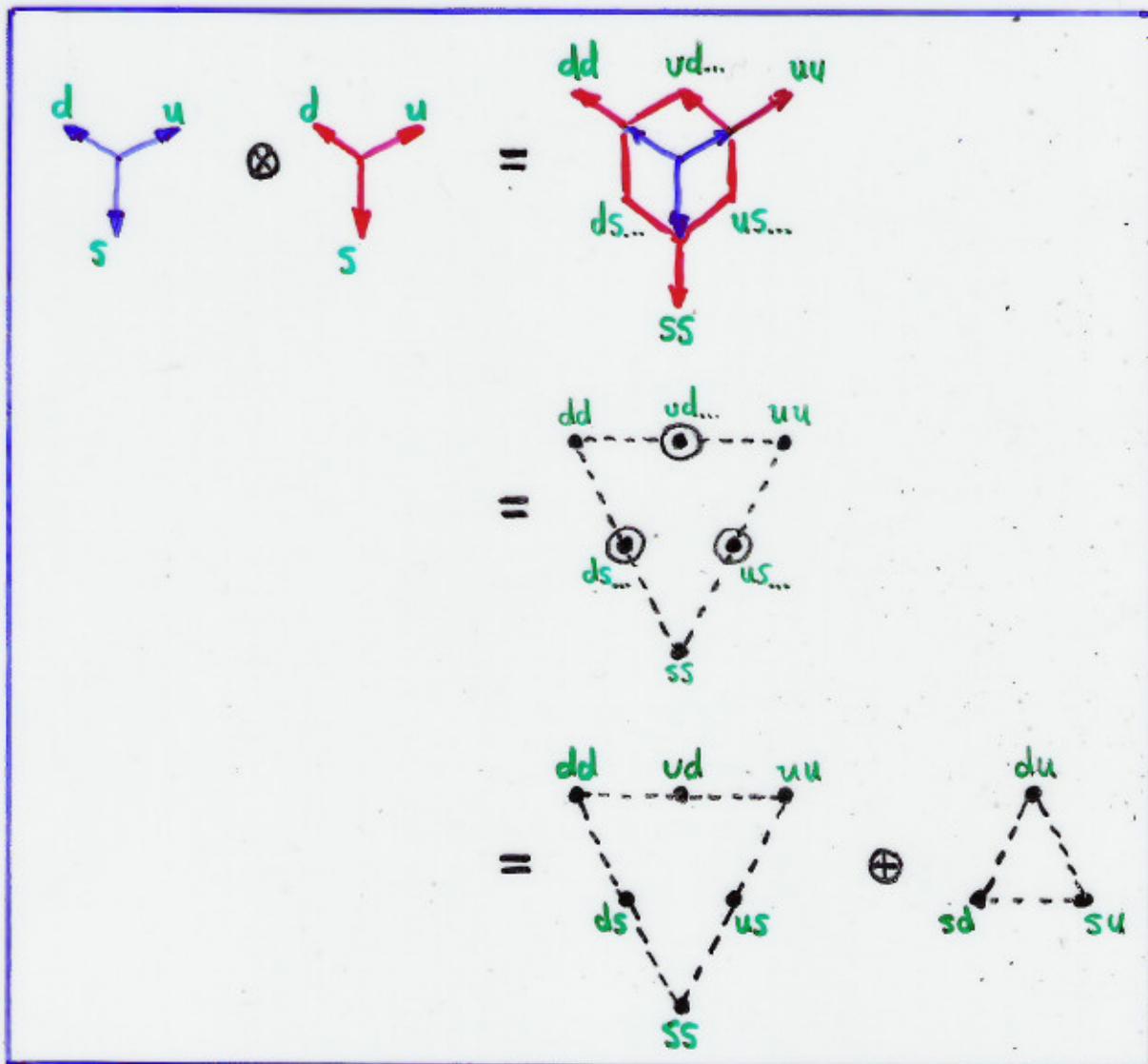


- Math:  $[3] \otimes [\bar{3}] = [8] \oplus [1]$
- Physics: mesons are not fundamental constituents of matter, rather they're composite particles. mesons owe their identity to the quarks that compose them.

(Remark: mesons are not "Baryons" thus the mesons should have  $B=0$  and they do because  $B_{\text{meson}} = B_q + B_{\bar{q}} = \frac{1}{3} - \frac{1}{3} = 0$ .)

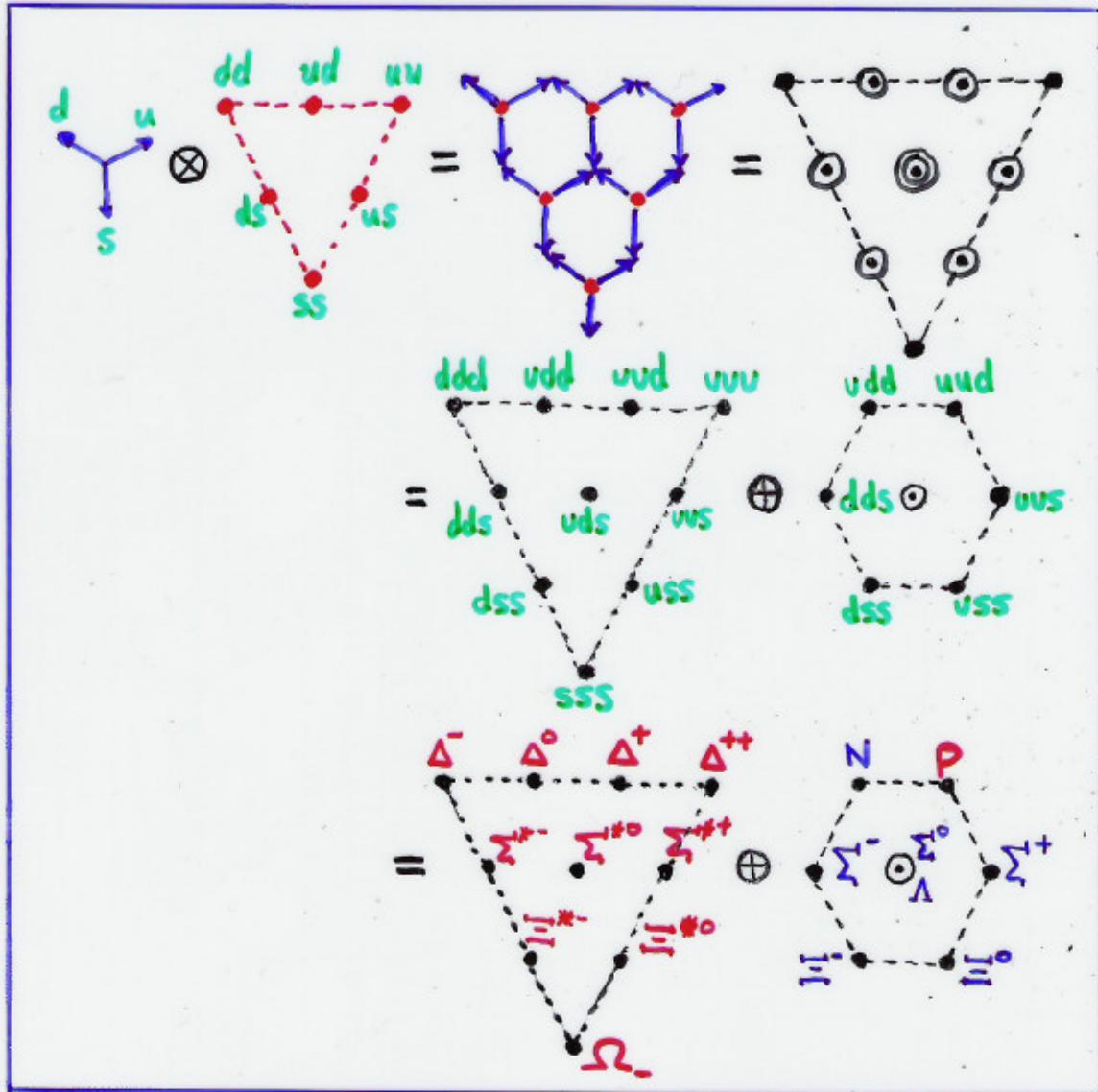
**MORE SU(3) FLAVOR:  $[3] \otimes [3] = [6] \oplus [\bar{3}]$**

- Put two quarks together, here's how:



## CONSTRUCTING BARYONS

Continuing the construction  $[3] \otimes [3] = [6] \oplus [\bar{3}]$  we add another quark and ignore the  $[\bar{3}]$  for now,



- Math:  $[3] \otimes [6] = [10] \oplus [8]$
- PHYSICS: BARYONS are made of 3 quarks.

(BARYONS HAVE  $B = B_{q_1} + B_{q_2} + B_{q_3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ )

## CONFESIONS AND CONCLUSIONS

- Quarks have spin  $\frac{1}{2}$ , thus are fermions, thus must obey the Pauli-Exclusion-Principal. We've ignored this so our results are schematic
- $[3]$  the quarks, and  $[\bar{3}]$  the antiquarks are fundamental rep. of **SU(3) FLAVOR** which is a global symmetry. **FLAVOR** symmetry put guidelines on the overall particle content of the theory, it has no dynamics.
- The 1<sup>st</sup> point above is solved by the introduction of COLOR SU(3). Each quark is assigned a color, **red, green, or blue** then SU(3) color is used to explain how the colored quarks interact. SU(3) COLOR is a local (gauge) symmetry which has a rich and complex structure (QCD)
- We've made no attempt to relate isospin to "weak isospin".
- It's fascinating how sucessful the quark-model has been, there is much more it explains then we have time for here. I hope you've seen enuf to understand why people think quarks exist. (well in as much as anything in a model exists...)