

## Differential Forms

Defn/ Let  $M$  be a manifold and  $p \in M$  then  $\gamma$  is a  $(^0_2)$  tensor on  $M$  iff  $\gamma$  is a function defined at points of  $M$  such that  $\forall q \in M \quad \gamma(q) = \gamma_q$  is a bilinear mapping from

$$T_q M \times T_q M \rightarrow \mathbb{R}$$

We will assume that  $\gamma$  is smooth in the following sense,

If  $\mathbf{x}$  and  $\mathbf{y}$  are smooth vector fields on  $M$  then the mapping from  $M$  to  $\mathbb{R}$  defined by

$$q \rightarrow \gamma(q)(\mathbf{x}(q), \mathbf{y}(q)) = \gamma_q(\mathbf{x}_q, \mathbf{y}_q)$$

Then  $\gamma$  would be smooth if the above were smooth  $\forall$  vector fields  $\mathbf{x}$  and  $\mathbf{y}$ . Once we see this in coordinates not so bad then, let  $(U, x)$  be chart,  $\mathbf{x}, \mathbf{y} \in C^\infty(U)$ .

$$\begin{aligned} \gamma(\mathbf{x}_q, \mathbf{y}_q) &= \gamma_q \left( \sum_i \mathbf{x}_x^i(q) \left( \frac{\partial}{\partial x^i}|_q \right), \sum_j \mathbf{y}_x^j(q) \left( \frac{\partial}{\partial x^j}|_q \right) \right) \\ &= \sum_i \mathbf{x}_x^i(q) \underbrace{\gamma_q \left( \frac{\partial}{\partial x^i}|_q, \sum_j \mathbf{y}_x^j(q) \frac{\partial}{\partial x^j}|_q \right)}_{\text{smooth on } U} \\ &= \sum_i \sum_j \mathbf{x}_x^i(q) \mathbf{y}_x^j(q) \underbrace{\gamma_q \left( \frac{\partial}{\partial x^i}|_q, \frac{\partial}{\partial x^j}|_q \right)}_{\text{smooth on } U} \\ &= \sum_i \sum_j \gamma_{ij}(q) \underbrace{\mathbf{x}_x^i(q) \mathbf{y}_x^j(q)}_{\text{linear comb. of smooth fcts. in } C^\infty U}. \end{aligned}$$

$\rightarrow \gamma$  smooth  $\Leftrightarrow \gamma_{ij}$  is smooth.

$\gamma_q(\mathbf{x}_q, \mathbf{y}_q) = ?$  what's a slicker formulation

$$\begin{aligned} d_q x^i(\mathbf{x}_q) &= d_q x^i \left( \sum_k \mathbf{x}_x^k(q) \left( \frac{\partial}{\partial x^k}|_q \right) \right) \\ &= \sum_k \mathbf{x}_x^k(q) d x^i \left( \frac{\partial}{\partial x^k}|_q \right) \\ &= \mathbf{x}_x^i(q) \end{aligned}$$

$$\therefore \gamma_q(\mathbf{x}_q, \mathbf{y}_q) = \sum_i \sum_j \gamma_{ij}(q) d_q x^i(\mathbf{x}_q) d_q y^j(\mathbf{y}_q)$$

$$d_q x^i(\mathbf{y}_q) = \mathbf{y}_x^i(q)$$

3/28/07

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TENSOR Product

$$\alpha, \beta \in \{T_q M \rightarrow \mathbb{R}\} = T_q^* M$$

$$(\alpha \otimes \beta)(\mathbf{x}, \mathbf{y}) = \alpha(\mathbf{x}) \beta(\mathbf{y})$$

So we can make  $\gamma_q$  even prettier

$$\gamma_q(\mathbf{x}_q, \mathbf{y}_q) = \sum_i \sum_j \gamma_{ij}(q) (d_q x^i \otimes d_q x^j)(\mathbf{x}_q, \mathbf{y}_q)$$

This holds for all  $\mathbf{x}_q$  and  $\mathbf{y}_q$  therefore we have a new animal

$$\boxed{\gamma_q = \sum_i \sum_j \gamma_{ij}(q) (d_q x^i \otimes d_q x^j)}$$

(A Basis of  $(T_x^0 M)_q$  is  $\{d_q x^i \otimes d_q x^j\}$ )

Now  $\gamma$  is a tensor of type  $(2, 0)$  if it is a smooth mapping on  $M$  such that  $\forall q \in M$  we have trilinear  $\gamma_q$  from,

$$\gamma_q : T_q M \times T_q M \times T_q M \rightarrow \mathbb{R}$$

$$(\alpha \otimes \beta \otimes \gamma)(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}) \beta(\mathbf{y}) \gamma(\mathbf{z})$$

$$\alpha, \beta, \gamma \in T_q^* M$$

$$\mu_q(\mathbf{x}_q, \mathbf{y}_q, \mathbf{z}_q) = \sum_{i,j,k} \mathbf{x}_x^i(q) \mathbf{y}_x^j(q) \mathbf{z}_x^k(q) \nu_q\left(\frac{\partial}{\partial x^i}|_q, \frac{\partial}{\partial x^j}|_q, \frac{\partial}{\partial x^k}|_q\right)$$

$$= \sum_{i,j,k} \mathbf{x}_x^i(q) \mathbf{y}_x^j(q) \mathbf{z}_x^k(q) \nu_{ijk}(q)$$

$$= \sum_{i,j,k} \nu_{ijk}(q) (d_q x^i \otimes d_q x^j \otimes d_q x^k)(\mathbf{x}_q, \mathbf{y}_q, \mathbf{z}_q)$$

$$\therefore \boxed{\nu = \sum_{i,j,k} \nu_{ijk} (d x^i \otimes d x^j \otimes d x^k)}$$

THE ALTERNATING ALGEBRA

Def<sup>n</sup>/ Let  $\gamma \in T_2^{\circ}M$ . We call  $\gamma$  skew-symmetric or alternating iff  $\gamma_{\bar{q}}(\bar{X}_{\bar{q}}, \bar{Y}_{\bar{q}}) = -\gamma_{\bar{q}}(\bar{Y}_{\bar{q}}, \bar{X}_{\bar{q}})$   $\forall \bar{q} \in M, \forall \bar{X}_{\bar{q}}, \bar{Y}_{\bar{q}} \in T_{\bar{q}}M$ .

Since  $\gamma \in T_2^{\circ}M$

$$\gamma = \sum_{i,j} \gamma_{ij} (dx^i \otimes dx^j)$$

$$\gamma_{k\ell} = \gamma\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right) = -\gamma\left(\frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k}\right) = -\gamma_{\ell k}$$

$$\boxed{\gamma \text{ skew} \iff \gamma_{k\ell} = -\gamma_{\ell k}}$$

$\gamma \in T_2^{\circ}M$  is alternating  $\iff \gamma_{ij}^x = -\gamma_{ji}^x \quad \forall i, j$  and charts  $x$ .

Def<sup>n</sup>/ Let  $\Lambda^2 M$  denote the set of all skew-symmetric elements of  $T_2^{\circ}M$ .

Operations on  $T_2^{\circ}M$ ,  $\bar{X}, \bar{Y} \in TM$

$$(\gamma_1 + \gamma_2)(\bar{X}, \bar{Y}) = \gamma_1(\bar{X}, \bar{Y}) + \gamma_2(\bar{X}, \bar{Y})$$

$$(c\gamma)(\bar{X}, \bar{Y}) = c\gamma(\bar{X}, \bar{Y})$$

Now  $T_2^{\circ}M$  is a vector space where  $\Lambda^2 M \leq T_2^{\circ}M$

Basis for the skew-symmetric objects  $\Lambda^2 M$ . 3/28/01

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Let  $\gamma \in \Lambda^2 M \subseteq T^* M$ .

$$\begin{aligned}\gamma &= \sum_i \sum_j \gamma_{ij} (dx^i \otimes dx^j) \\ &= \sum_{i < j} \gamma_{ij} (dx^i \otimes dx^j) + \sum_{j < i} \gamma_{ij} (dx^i \otimes dx^j) + \cancel{\sum_{i=j} \gamma_{ii} (dx^i \otimes dx^i)} \\ &= \sum_{i < j} \gamma_{ij} (dx^i \otimes dx^j) + \sum_{i < j} \gamma_{ji} (dx^j \otimes dx^i) \quad \gamma_{ij} = -\gamma_{ji} \therefore \gamma_{ii} = 0 \\ &= \sum_{i < j} \gamma_{ij} (dx^i \otimes dx^j) + \sum_{i < j} -\gamma_{ij} (dx^j \otimes dx^i) \\ &= \sum_{i < j} \gamma_{ij} (dx^i \otimes dx^j - dx^j \otimes dx^i)\end{aligned}$$

$$= \sum_{i < j} 2\gamma_{ij} (dx^i \wedge dx^j)$$

Def<sup>n</sup>/ The wedge product on two oneforms is just

$$dx \wedge dy = \frac{1}{2}[dx \otimes dy - dy \otimes dx]$$

Notice that  $\{dx^i \wedge dx^j\}_{i < j}$  spans  $\Lambda^2 M$ .

These are linearly independent,  $\{dx^i \wedge dx^j\}_{i < j}$  is a basis for  $\Lambda^2 M$ .

$$\begin{aligned}(dx^i \wedge dx^j)(X, Y) &= \frac{1}{2}(X^i Y^j - X^j Y^i) \\ &= -\frac{1}{2}(X^j Y^i - X^i Y^j) \\ &= -(dx^i \wedge dx^j)(Y, X)\end{aligned}$$

But also we get,

$$\begin{aligned}(dx^i \wedge dx^j)(X, Y) &= \frac{1}{2}(X^i Y^j - Y^i X^j) \\ &= -(dx^j \wedge dx^i)(X, Y)\end{aligned}$$

$$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

general property  
of wedge  
product

$\Lambda^2 M$  is a Grassmann Algebra.

① 4/2/2002

Define/ Wedge product of  $dx^i$  where  $\langle dx^i \rangle_m^{i=1} = T_u^* M$ ,  $i \in \{1, \dots, m\}$

$$dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} = \sum_{\sigma \in \Sigma_p} \text{sgn}(\sigma) (dx^{i_{\sigma(1)}} \otimes dx^{i_{\sigma(2)}} \otimes \dots \otimes dx^{i_{\sigma(p)}})$$

Then  $(dx^{i_1} \wedge \dots \wedge dx^{i_p})$  is a  $p$ -multilinear mapping. It is skew symmetric in the sense that

$$b\left(\frac{\partial}{\partial x^{j_1}(x)}, \frac{\partial}{\partial x^{j_2}(x)}, \dots, \frac{\partial}{\partial x^{j_p}(x)}\right) = \text{sgn}(\gamma) b\left(\frac{\partial}{\partial x^{j_1}}, \frac{\partial}{\partial x^{j_2}}, \dots, \frac{\partial}{\partial x^{j_p}}\right)$$

(an note sgn( $\gamma$ ) by  $(-1)^{\gamma}$  where  $\gamma$  is taken to be even if  $\gamma$  is even product of transpositions while  $\gamma$  is odd if it is an odd product of transpositions.)

We must restrict  $i$  and  $j$  if  $\{dx^i \wedge dx^j\}$  is to make a basis for  $\Lambda^p T_u^* M = (\Delta^p M)_u$  ( $p=2$ )

$$1(dx^i \wedge dx^j) + 1(dx^j \wedge dx^i) = 0$$

So if  $\gamma$  is  $p$ -multilinear and skew then (restriction is  $i < j$ ).

$$\gamma_u = \sum_{j_1 < j_2 < \dots < j_p} \gamma_{j_1 \dots j_p}(u) (dx^{j_1} \wedge \dots \wedge dx^{j_p})$$

since skew  
 $\downarrow$

Define nonzero  $\gamma_{j_1 \dots j_p}$  by  $\gamma_{j_1 \dots j_p} = \text{sgn}(\sigma) \gamma_{j_{\sigma(1)} \dots j_{\sigma(p)}}$  ( $\gamma_{ii} = 0$ )

Thus,

$$\gamma = \frac{1}{p!} \sum_{j_1} \sum_{j_2} \dots \sum_{j_p} \gamma_{j_1 \dots j_p} (dx^{j_1} \wedge \dots \wedge dx^{j_p})$$

Physics Notation

$$\gamma = \frac{1}{p!} \gamma_{j_1 \dots j_p} (dx^{j_1} \wedge \dots \wedge dx^{j_p})$$

## Exterior Derivative on Forms

$$\gamma = \sum_{j_1 < j_2 < \dots < j_p} \gamma_{j_1 \dots j_p} (dx^{j_1} \wedge \dots \wedge dx^{j_p})$$

$$d\gamma = \sum_{j_1 < j_2 < \dots < j_p} (d\gamma_{j_1 \dots j_p}) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

$$= \sum_{j_1 < j_2 < \dots < j_p} \sum_i \frac{\partial \gamma_{j_1 \dots j_p}}{\partial x^i} (dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p})$$

$$= \gamma_{[j_1 \dots j_p, i]} (dx^{j_1} \wedge \dots \wedge dx^{j_p} \wedge dx^i)$$

↑  
antisymmetrizing [ ]

and differentiating ,

(Notation of MTW).

Can be shown that  $d(d\gamma) = 0$ , will be done  
rigorously in Manifolds

Example Take  $M = \mathbb{R}^3$  we generate the vector calculus from  
this formalism,

0 - form  $f$

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \Rightarrow \boxed{df = \nabla f}$$

1 - form  $\alpha$

$$\alpha = \sum_i \alpha_i dx^i = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3$$

$$d\alpha = d\alpha_1 \wedge dx + d\alpha_2 \wedge dy + d\alpha_3 \wedge dz$$

$$d\alpha = \left( \frac{\partial \alpha_1}{\partial x} dx + \frac{\partial \alpha_2}{\partial y} dy + \frac{\partial \alpha_3}{\partial z} dz \right) \wedge dx + \left( \frac{\partial \alpha_1}{\partial y} dx + \frac{\partial \alpha_2}{\partial x} dy + \frac{\partial \alpha_3}{\partial z} dz \right) \wedge dy$$

$$+ \left( \frac{\partial \alpha_1}{\partial z} dx + \frac{\partial \alpha_3}{\partial y} dy + \frac{\partial \alpha_2}{\partial z} dz \right) \wedge dz$$

$$= \left( -\frac{\partial \alpha_1}{\partial y} + \frac{\partial \alpha_2}{\partial x} \right) (dx \wedge dy) + \left( \frac{\partial \alpha_1}{\partial z} - \frac{\partial \alpha_3}{\partial x} \right) dz \wedge dx + \left( \frac{\partial \alpha_2}{\partial z} + \frac{\partial \alpha_3}{\partial y} \right) dy \wedge dz$$

Now if we consider the usual formalism with  $\vec{\alpha}$  a vector field,

$$\vec{\alpha} = \alpha_1 \hat{i} + \alpha_2 \hat{j} + \alpha_3 \hat{k}$$

We identify  $\hat{i} \hat{j} \hat{k}$  with  $\hat{i} = \hat{j} \times \hat{k} \sim dy \wedge dz$  and also

$$\hat{j} = \hat{k} \times \hat{i} \sim dz \wedge dx \text{ and } \hat{k} = \hat{i} \times \hat{j} \sim dx \wedge dy$$

So we may identify the coefficients of  $d\alpha$  with the curl of  $\vec{\alpha}$

$$\text{curl}(\vec{\alpha}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} = \hat{i} \left( \frac{\partial \alpha_3}{\partial y} - \frac{\partial \alpha_2}{\partial z} \right) - \hat{j} \left( \frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial z} \right) + \hat{k} \left( \frac{\partial \alpha_1}{\partial x} - \frac{\partial \alpha_2}{\partial y} \right)$$

$$\therefore d\alpha = (\text{curl}(\vec{\alpha}))_1 (dy \wedge dz) + (\text{curl}(\vec{\alpha}))_2 dz \wedge dx + (\text{curl}(\vec{\alpha}))_3 dx \wedge dy$$

So the exterior derivative of a one form is the curl.

A general 2-form can be expressed in basis  $dx \wedge dy, dz \wedge dx, dy \wedge dz$

$$\beta = \beta_1 (dy \wedge dz) + \beta_2 (dz \wedge dx) + \beta_3 (dy \wedge dx)$$

$$d\beta = \partial \beta_1 \wedge dy \wedge dz + \partial \beta_2 \wedge dz \wedge dx + \partial \beta_3 \wedge dy \wedge dx$$

$$= \left( \frac{\partial \beta_1}{\partial x} dx + \frac{\partial \beta_2}{\partial y} dy + \frac{\partial \beta_3}{\partial z} dz \right) \wedge dy \wedge dz + \partial \beta_2 \wedge dz \wedge dx + \partial \beta_3 \wedge dy \wedge dx$$

$$= \frac{\partial \beta_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial \beta_2}{\partial y} \cancel{dx \wedge dz} + \frac{\partial \beta_3}{\partial z} dz \wedge dy \wedge dx$$

$$= \left( \frac{\partial \beta_1}{\partial x} + \frac{\partial \beta_2}{\partial y} + \frac{\partial \beta_3}{\partial z} \right) dx \wedge dy \wedge dz$$

So if we construct vector field  $\vec{\beta} = \beta_1 \hat{i} + \beta_2 \hat{j} + \beta_3 \hat{k}$

$$d\beta = (\text{Div } \vec{\beta})(dx \wedge dy \wedge dz)$$

$$3\text{-form} - \gamma = f(dx \wedge dy \wedge dz)$$

$$d\gamma = df \wedge (dx \wedge dy \wedge dz)$$

$$= (f_x dx + f_y dy + f_z dz) \wedge dx \wedge dy \wedge dz = 0$$

because  $x, y, z$  must be repeated and  $\Rightarrow d\gamma = 0$ .

## Electro Magnetism in the language of Differential Forms

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

$$\nabla \cdot \vec{E} = \rho$$

Maxwell's  
equation's

Introduce the Faraday which we hope to get 2 form from.

$$(F_{\mu\nu}) = \begin{bmatrix} t & x & y & z \\ 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

$$dF = j = (\vec{j}, \rho)$$

$$d^* F = 0$$

$$F = \frac{1}{2} \sum_{\mu<\nu} F_{\mu\nu} (dx^\mu \wedge dx^\nu)$$

Where we take  $x^0 = ct$ ,  $x^1 = x$ ,  $x^2 = y$ ,  $x^3 = z$

$$F = \sum_{\mu<\nu} F_{\mu\nu} (dx^\mu \wedge dx^\nu)$$

$$= \sum_{i=1}^3 F_{0i} (dx^0 \wedge dx^i)$$

$$\textcircled{I} = -E_1(c dt \wedge dx) + (-E_2)[c dt \wedge dy] + (-E_3)[c dt \wedge dz]$$

$$= c(E_1 dx + E_2 dy + E_3 dz) \wedge dt$$

$$\textcircled{II} = \sum_{1< j} F_{ij} (dx^i \wedge dx^j) = B_3(dx \wedge dy) - B_2(dx \wedge dz)$$

$$\textcircled{III} = \sum_{2< j} F_{2j} (dx^2 \wedge dx^j) = F_{23} dy \wedge dz = B_1 dy \wedge dz$$

$$\textcircled{I}, \textcircled{II}, \textcircled{III} \Rightarrow F = -E \wedge dt + B \quad \leftarrow \begin{aligned} B &= B_1 dy \wedge dz + B_2 dx \wedge dz \\ &\quad + B_3 dx \wedge dy \end{aligned}$$

$$E = E_1 dx + E_2 dy + E_3 dz$$

## Properties of Differential Forms

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

$$dd\gamma = d^2\gamma = 0$$

0-form	$f$	$df = f_x dx + f_y dy + f_z dz$	$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$
1-form	$\alpha$	$d\alpha = (\text{curl } \vec{\alpha})_1 (dy \wedge dz) + (\text{curl } \vec{\alpha})_2 (dz \wedge dx) + (\text{curl } \vec{\alpha})_3 (dx \wedge dy)$	$\vec{\alpha} = \alpha_1 \hat{i} + \alpha_2 \hat{j} + \alpha_3 \hat{k}$
2-form	$\beta$	$d\beta = (\text{Div } \vec{\beta})(dx \wedge dy \wedge dz)$	$\vec{\beta} = \beta_1 \hat{i} + \beta_2 \hat{j} + \beta_3 \hat{k}$
3-form	$\gamma$	$d\gamma = 0$	$\beta = \beta_1 (dz \wedge dx) + \beta_2 (dz \wedge dx) + \beta_3 (dx \wedge dy)$ $dx \wedge dy \wedge dz$ ~ volume element.

General Expansion of  $k$ -form;

$$\gamma_k = \sum_{i_1 < i_2 < \dots < i_k} \gamma_{i_1 i_2 \dots i_k} (dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k})$$

Exterior Derivative of  $k$ -form

$$d\gamma = \sum_{i_1 < i_2 < \dots < i_k} (d\gamma_{i_1 \dots i_k}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Multi-index

$$I = (i_1, i_2, \dots, i_k) \quad \text{increasing } i_j$$

$$\gamma = \sum_I \gamma_I dx^I$$

$$d\gamma = \sum_{I, J} \left( \frac{\partial \gamma_I}{\partial x^J} \right) (dx^J \wedge dx^I)$$

## FARADAY TENSOR

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

$$E = E_1 dx + E_2 dy + E_3 dz$$

$$B = B_1 (dy \wedge dz) + B_2 (dz \wedge dx) + B_3 (dx \wedge dy)$$

$$F = \frac{1}{2} \sum_{\mu, \nu} F_{\mu\nu} (dx^\mu \wedge dx^\nu), \quad \cancel{\partial^2 = \partial(\partial x \wedge \partial y)}$$

$$x^\mu = (t, x, y, z)$$

So the  $E$  field is a one form, the  $B$  field is a two form and the Faraday is a two form. Maxwell's Equations in terms of the vector fields  $\vec{E}$  and  $\vec{B}$  where  $E_i, B_i : \mathbb{R}^4 \rightarrow \mathbb{R}$ . So if you wish  $\vec{E} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  and  $\vec{B} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  then

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}, \quad \nabla \cdot \vec{E} = \rho$$

$$\vec{E} = E_1 \hat{i} + E_2 \hat{j} + E_3 \hat{k}$$

$$\vec{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$$

Roughly we write  $\vec{E}$  as one form since it is natural to inquire about the work done along line while  $\vec{B} \rightarrow 2$  form because often we are interested in flux of  $\vec{B}$  through surface. Now last class we showed that  $F = E \wedge dt + B$

$$dF = d(E \wedge dt) + dB$$

$$= dE \wedge dt + (-1)^1 E \wedge d(dt) + dB$$

$$= dE \wedge dt + dB$$

$$= [(\text{curl } \vec{E})_1 (dy \wedge dz) + (\text{curl } \vec{E})_2 (dz \wedge dx) + (\text{curl } \vec{E})_3 (dx \wedge dy)] \wedge dt$$

$$+ \left[ \frac{\partial E_1}{\partial t} dt + \frac{\partial E_2}{\partial t} dt + \frac{\partial E_3}{\partial t} dt \right] \wedge dt + (\text{div } \vec{B}) dx \wedge dy \wedge dz$$

$$+ \left[ \frac{\partial B_1}{\partial t} (dt \wedge dy \wedge dz) + \frac{\partial B_2}{\partial t} (dt \wedge dz \wedge dx) + \frac{\partial B_3}{\partial t} (dt \wedge dx \wedge dy) \right]$$

(2)

$$dF = \left[ -\frac{\partial B_1}{\partial t} (dy \wedge dz) \wedge dt - \frac{\partial B_2}{\partial t} (dz \wedge dx) \wedge dt - \frac{\partial B_3}{\partial t} (dx \wedge dy) \wedge dt \right]$$

$$+ \left[ \frac{\partial B_1}{\partial t} (dt \wedge dy \wedge dz) + \frac{\partial B_2}{\partial t} (dt \wedge dz \wedge dx) + \frac{\partial B_3}{\partial t} (dt \wedge dx \wedge dy) \right]$$

$\therefore dF = 0$  this is synonymous with  $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$  and  $\nabla \cdot \vec{B} = 0$

Notice that conversely  $dF = 0 \Rightarrow$  maxwell's  $E_3 = \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ,  $\nabla \cdot \vec{B} = 0$

$$dF = \left[ (\nabla \times \vec{E})_1 + \frac{\partial E_3}{\partial t} \right] dy \wedge dz \wedge dt + \left[ (\nabla \times \vec{E})_2 + \frac{\partial E_1}{\partial t} \right] dz \wedge dx \wedge dt$$

$$+ \left[ (\nabla \times \vec{E})_3 + \frac{\partial E_2}{\partial t} \right] (dx \wedge dy \wedge dt) + [\operatorname{div} \vec{B}] (dx \wedge dy \wedge dz)$$

Now these terms are spanned by LI wedges  $dx \wedge dy \wedge dz$  ...  
 Thus  $dF = 0 \Leftrightarrow$  coefficients are zero  $\Leftrightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ,  $\nabla \cdot \vec{B} = 0$

A (could be proven, here we define by  $E \leftrightarrow B$ )

THE HODGE DUAL

$$(*F)_{\mu\nu} = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix}$$

this is the  
"Maxwell" if it is  
Dual to the Faraday

$$*F = \frac{1}{2} \sum_{\mu, \nu} (*F)_{\mu\nu} (dx^\mu \wedge dx^\nu)$$

$$4-2-1 = 1$$

$$4-1-1 = 2$$

$$*B = \sum_i B_i dx^i$$

$$*E = E_1 (dy \wedge dz) + E_2 (dz \wedge dx) + E_3 (dx \wedge dy)$$

$$d(*F) = \left[ (\nabla \times \vec{B})_1 - \frac{\partial E_1}{\partial t} \right] (dt \wedge dy \wedge dz) + \left[ (\nabla \times \vec{B})_2 - \frac{\partial E_2}{\partial t} \right] (dt \wedge dz \wedge dx)$$

$$+ \left[ (\nabla \times \vec{B})_3 - \frac{\partial E_3}{\partial t} \right] (dt \wedge dx \wedge dy) + [\nabla \cdot \vec{E}] (dx \wedge dy \wedge dz)$$

$$j = [j_1(dy \wedge dz) + j_2(dt \wedge dx) + j_3(dx \wedge dy)] \wedge dt$$

$$d^* F = j_1(dt \wedge dy \wedge dz) + j_2(dt \wedge dz \wedge dx) + j_3(dt \wedge dx \wedge dy) + P(dx \wedge dy \wedge dt)$$

$\underbrace{\hspace{10em}}$   
J

$$\boxed{d^* F = J} \quad \leftarrow \nabla \times \vec{E} = -\frac{\partial \vec{E}}{\partial t}, \quad \nabla \cdot \vec{E} = P$$

F curvature on principle fibre bundle, postulate E and Q fields make up curvature of connections

$$\boxed{dF = 0} \quad \Longleftrightarrow \quad d^* F = dA \quad \begin{matrix} \text{exact} \Rightarrow \text{closed} \\ \text{on convex set} \end{matrix}$$

$$\boxed{d^* F = 0} \quad \Longleftrightarrow \quad d^* dA = 0$$

Now then, A is made of functions on  $\mathbb{R}^4$

$$A = A_0 dt + A_1 dx + A_2 dy + A_3 dz$$

$$\vec{A} = (A_1, A_2, A_3) \quad \leftarrow \text{vector potential}$$

$$dA$$

$$dt$$

$$dx$$

$$dy$$

$$dz$$

$$dt$$

$\mathbb{R}^3$ , Euclidean norm

If  $e_1, e_2, e_3$  is the standard orthonormal basis of  $\mathbb{R}^3$  then

$$*e^1 = e^2 \wedge e^3, *e^2 = e^3 \wedge e^1, *e^3 = e^1 \wedge e^2$$

Thus

$$\begin{cases} *dx = dy \wedge dz \\ *dy = dz \wedge dx \\ *dz = dx \wedge dy \end{cases}$$

$\mathbb{R}^4$ , Minkowski "norm".

If  $e_0, e_1, e_2, e_3$  is the standard basis of  $\mathbb{R}^4$  orthonormal relative to the Minkowski metric then

$$*e^0 = e^1 \wedge e^2 \wedge e^3$$

$$*e^1 = e^0 \wedge e^2 \wedge e^3$$

$$*e^2 = -e^0 \wedge e^1 \wedge e^3 = e^0 \wedge e^3 \wedge e^1$$

$$*e^3 = e^0 \wedge e^1 \wedge e^2$$

$$*(e^1 \wedge e^3) = -e^0 \wedge e^2$$

$$*(e^0 \wedge e^1 \wedge e^3) = -e^2$$

$$\begin{cases} *dt = dx \wedge dy \wedge dz \\ *dx = dt \wedge dy \wedge dz \\ *dy = dt \wedge dz \wedge dx \\ *dz = dt \wedge dx \wedge dy \end{cases}$$

The proof is an exercise in Curtis + Miller's Differentiable Manifolds and Theoretical Physics



Let  $(x^\mu) = (t, x, y, z)$  and

$$({}^4F_{\mu\nu}) = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}.$$

$$\begin{aligned} \text{Then } {}^4F &= \sum_{\mu < \nu} ({}^4F)_{\mu\nu} (dx^\mu \wedge dx^\nu) \\ &= \sum_{i=1}^3 ({}^4F)_{0i} (dt \wedge dx^i) + \sum_{i=2}^3 ({}^4F)_{ii} (dx^i \wedge dx^i) + {}^4F_{23} (dx^2 \wedge dx^3) \\ &= B_1 (dt \wedge dx) + B_2 (dt \wedge dy) + B_3 (dt \wedge dz) \\ &\quad + E_3 (dx \wedge dy) + (-E_2) (dx \wedge dz) + E_1 (dy \wedge dz) \end{aligned}$$

$$\begin{aligned} {}^3(B) &= {}^3[B_1(dy \wedge dz) + B_2(dz \wedge dx) + B_3(dx \wedge dy)] \\ &= B_1 dx + B_2 dy + B_3 dz \end{aligned}$$

$$\begin{aligned} {}^3E &= {}^3(E_1 dx + E_2 dy + E_3 dz) \\ &= E_1 (dy \wedge dz) + E_2 (dz \wedge dx) + E_3 (dx \wedge dy) \end{aligned}$$

$$\begin{aligned} \text{So } {}^4F &= dt \wedge (B_1 dx + B_2 dy + B_3 dz) + E_1 (dy \wedge dz) + \\ &\quad + E_2 (dz \wedge dx) + E_3 (dx \wedge dy) \end{aligned}$$

$$\text{and } {}^4F = dt \wedge ({}^3B) + ({}^3E)$$

$$\begin{aligned} d({}^4F) &= -d[({}^3B) \wedge dt] + d({}^3E) \\ &= -d({}^3B) \wedge dt + d({}^3E) \end{aligned}$$



$$\begin{aligned}
d(*^3 B) &= d(B_1 dx + B_2 dy + B_3 dz) \\
&= (\text{curl } \vec{B})_1 (dy \wedge dz) + (\text{curl } \vec{B})_2 (dz \wedge dx) + (\text{curl } \vec{B})_3 (dx \wedge dy) \\
&\quad + \frac{\partial B_1}{\partial t} (dt \wedge dx) + \frac{\partial B_2}{\partial t} (dt \wedge dy) + \frac{\partial B_3}{\partial t} (dt \wedge dz) \\
d(*^3 E) &= (\text{Div } \vec{E}) (dx \wedge dy \wedge dz) \\
&\quad + \frac{\partial E_1}{\partial t} (dt \wedge dy \wedge dz) + \frac{\partial E_2}{\partial t} (dt \wedge dz \wedge dx) \\
&\quad + \frac{\partial E_3}{\partial t} (dt \wedge dx \wedge dy)
\end{aligned}$$

$$\begin{aligned}
d(*^4 F) &= -dt \wedge d(*^3 B) + d(*^3 E) \\
&= \left( -(\text{curl } \vec{B})_1 + \frac{\partial E_1}{\partial t} \right) (dt \wedge dy \wedge dz) \\
&\quad + \left( -(\text{curl } \vec{B})_2 + \frac{\partial E_2}{\partial t} \right) (dt \wedge dz \wedge dx) \\
&\quad + \left( -(\text{curl } \vec{B})_3 + \frac{\partial E_3}{\partial t} \right) (dt \wedge dx \wedge dy) \\
&\quad + \frac{\partial E_1}{\partial t} (dt \wedge dy \wedge dz) + \frac{\partial E_2}{\partial t} (dt \wedge dz \wedge dx) \\
&\quad + \frac{\partial E_3}{\partial t} (dt \wedge dx \wedge dy) + (\text{Div } \vec{E}) (dx \wedge dy \wedge dz)
\end{aligned}$$

$$\text{def } J = -\vec{J}_1 (dt \wedge dy \wedge dz) - \vec{J}_2 (dt \wedge dz \wedge dx) - \vec{J}_3 (dt \wedge dx \wedge dy) \\
+ \rho (dx \wedge dy \wedge dz)$$

Then  $d(*^4 F) = J$  if and only if

$$\text{curl } \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \quad \text{and} \quad \text{Div } \vec{E} = \rho$$



1.) Handout describing Dual operator  $\ast$  on  $\mathbb{R}^3$  and  $\mathbb{R}^4$  4/9/01

2.) Handout discussion of  $d(\ast F) = J$  being 2 of the 4 maxwell eq<sup>1</sup> recall  
 $dF = 0$  gives the other 2.

3.) Exercise on  $\Sigma + M$  Lagrangian  
needs connection  $\lambda \cdot A = (\lambda^{-1})^* A$  in 4(c)

Some Comments on Exercise 4

$$x^\mu = (t, x, y, z) = (x^0, x^1, x^2, x^3)$$

(a)  $\lambda : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is a Lorentz transformation iff

$$\hat{\eta}(\lambda(x), \lambda(y)) = \hat{\eta}(x, y) \quad \forall x, y$$

Let  $\Lambda$  be matrix such that

$$\lambda^i(x) = x^j \Lambda_j^i$$

$$\lambda(x) = x \Lambda$$

Then note,

$$\begin{aligned} \hat{\eta}(\lambda(x), \lambda(y)) &\equiv x \Lambda \eta(y \Lambda)^t \\ &= x \underbrace{\Lambda \eta \Lambda^T}_{\eta} y^T = x \eta y^T = \hat{\eta}(x, y) \end{aligned} \quad \forall x, y$$

$$(\Lambda \eta \Lambda^T)_b^a = \Lambda_k^a \eta_l^k \underbrace{\eta(l)}_b^l = \eta_b^a \leftarrow \text{row} \quad \leftarrow \text{column}$$

$$\boxed{\Lambda_k^a \eta^{kl} \Lambda_l^b = \eta_b^a \Rightarrow \hat{\eta}(\lambda(x), \lambda(y)) = \hat{\eta}(x, y)}$$

Exercise #4 : Hints or discussion I'm not sure which.

a)  $\alpha = \frac{1}{2} \alpha_{ij} (dx^i \wedge dx^j)$      $1 \leq i, j \leq 3$

$\beta = \frac{1}{2} \beta_{kl} (dx^k \wedge dx^l)$

$$\tilde{\eta}(\alpha, \beta) = \eta^{ik} \eta^{jl} \alpha_{ij} \beta_{kl} = \alpha^{kl} \beta_{kl}$$

Non-degenerate means  $\tilde{\eta}(\alpha, \beta) = 0 \Rightarrow \alpha = 0 = \beta$

Hint: Convert  $\tilde{\eta}(\alpha, \beta) = \eta^{ik} \eta^{jl} \alpha_{ij} \beta_{kl}$  into Moyal type notation we get

$$\text{Trace} [\hat{\eta} \hat{\alpha} \hat{\eta} \hat{\beta}] = 0$$

$$\text{Trace} [A \hat{\beta}] = 0 \quad (\text{choose } A = \hat{\beta}^T)$$

$$\text{Trace} [\hat{\beta}^T \hat{\beta}] = 0 \Rightarrow \hat{\beta} = 0.$$

### Remark

What is  $\lambda^i(x)$ ?

$$\lambda^i(x) = x^i \Delta_j^i \quad \text{or perhaps } \lambda^i(p) = x^i(p) \Delta_j^i$$

So  $x$  is a mapping while  $p$  is a fixed point.

$$\lambda^i(p) = x^i(p) \Delta_j^i$$

$$\lambda^i = \Delta_j^i x^i$$

$$d\lambda^i = \Delta_j^i dx^i$$

Why all this well he is getting ready to explain the pullback.

$$d\lambda^i(\partial_k) = (\Delta_j^i dx^i)(\partial_k) = \Delta_k^i$$

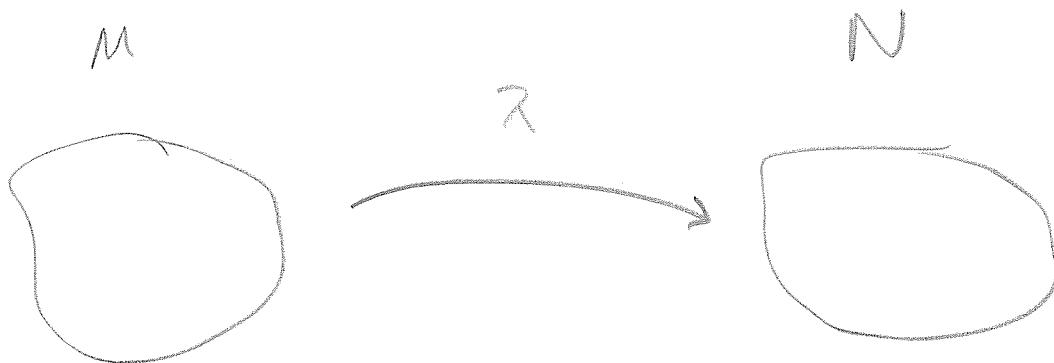
$$(dx^i)(\partial_k) = (\Delta_j^i dx^i)(\partial_k) = \Delta_k^i = \Delta^i_k$$

## Pullback

$$\begin{aligned} (\lambda^* \alpha)(v, w) &= \alpha(d\lambda(v), d\lambda(w)) \\ &= \frac{1}{2} \alpha_{ij} (dx^i \otimes dx^j)(d\lambda(v), d\lambda(w)) \end{aligned}$$

Notice that

$$\begin{aligned} (dx^i \otimes dx^j)(d\lambda(v), d\lambda(w)) &= dx^i(d\lambda(v)) dx^j(d\lambda(w)) \\ &= d(x^i \circ \lambda)(v) d(x^j \circ \lambda)(w) \\ x^i \circ \lambda &= \lambda^i \end{aligned}$$



$\lambda^* \alpha$  is a p form  
on M

$\alpha$  is p form  
on N

the  $\alpha$ -helix is the most stable conformation of the polypeptide chain.

The  $\beta$ -sheet is the second most stable conformation of the polypeptide chain.

$\beta$ -sheet

the  $\beta$ -sheet is a secondary structure of protein in which the backbone atoms of one peptide chain are involved in hydrogen bonding with the backbone atoms of another peptide chain.

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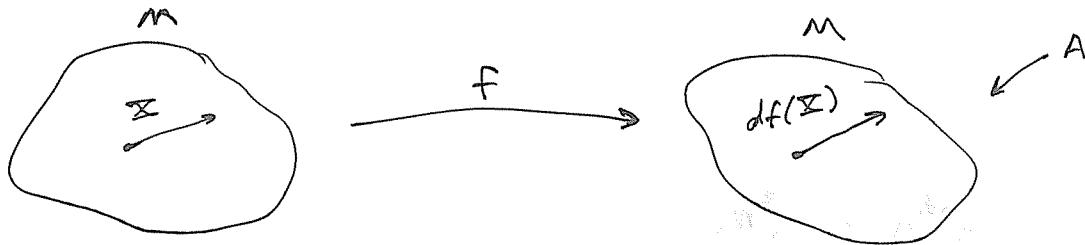
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## Pull back

$$f : M \longrightarrow N$$

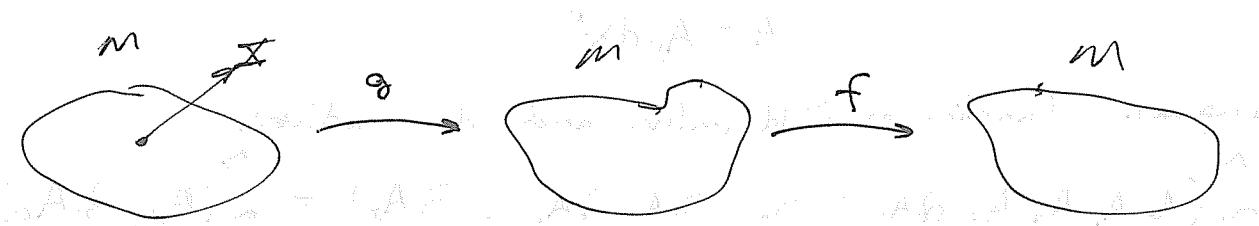
$$f^* A = A \circ df \quad \text{Let } A \text{ be}$$



$$(f^* A)(x) = A(df(x))$$

~~$$A^* df = f^* A^*$$~~

$$(f^* \alpha)(x, y) = \alpha(df(x), df(y))$$



$$[(f \circ g)^* A](x) = A(d(f \circ g)(x))$$

$$= A(df(dg(x)))$$

$$= (f^* A)(dg(x))$$

$$= g^*(f^* A)(x) \quad \forall x$$

~~$$\therefore (f \circ g)^* A = g^*(f^* A)$$~~

4/15/2001

$$f : M \rightarrow N$$

(2)

part 3  
 make  $\lambda \cdot A = (\lambda^{-1})^* A$

$$f^* : \Lambda^k N \rightarrow \Lambda^k M ; \text{ pullback}$$

$$df : TM \rightarrow TN ; \text{ differential}$$

For Example,

$$f : M \rightarrow N$$

$$\alpha \in \Lambda^2 N \Rightarrow f^* \alpha \in \Lambda^2 M$$

Group action pullback covectors, which is why prob 3 was wrong we want the relation boxed above.

- VECTOR POTENTIAL : Real valued 1-form on Minkowski Space

$$A = A_\mu dx^\mu$$

- Lagrangian ; Function of field values and derivatives.

$$\hat{\mathcal{L}}(A_0, A_1, A_2, A_3, \partial_0 A_0, \partial_1 A_0, \dots, \partial_3 A_0, \partial_0 A_1, \dots, \partial_3 A_3) = \hat{\mathcal{L}}(A_\mu, \partial_\nu A_\rho)$$

$$\hat{\mathcal{L}}(A_\mu, \partial_\nu A_\rho) = \int_{\text{Minkowski}} \tilde{\eta}(F_A, F_A) d^4x - (x)(A^*(e^+))$$

$$F_A = dA = d(A_\mu dx^\mu) = \frac{1}{2} (\partial_\nu A_\mu - \partial_\mu A_\nu)(dx^\nu \wedge dx^\mu)$$

Plugging into formula for what  $\tilde{\eta}$  means switching  $i, j \rightarrow \mu, \nu$ 's

$$\text{Action} = \hat{\mathcal{L}}(A_\mu, \partial_\nu A_\rho) = \frac{1}{4} \left[ \eta^{\mu\nu} \eta^{\nu\rho} (\partial_\nu A_\mu - \partial_\mu A_\nu)(\partial_\rho A_\nu - \partial_\nu A_\rho) \right] d^4x$$

$$\mathcal{L} = \frac{1}{4} \eta^{\nu\rho} \eta^{\mu\nu} (\partial_\nu A_\mu - \partial_\mu A_\nu)(\partial_\rho A_\nu - \partial_\nu A_\rho)$$

$$S(A_\mu) = \int \mathcal{L}(A_\mu, \partial_\nu A_\mu) d^4x = \text{Action}$$

Lagrangian : function of fields. A field is typically a function from Minkowski space, or at least some space with a Lorentzian Metric, ie a curved manifold that is Locally  $\cong$  Minkowski. Specifically

$$\begin{aligned} g : M &\rightarrow T^*M \\ \hookrightarrow g_p : T_p M \times T_p M &\rightarrow \mathbb{R} \end{aligned} \quad \left. \begin{array}{l} \text{Lorentzian} \\ \text{Metric} \\ \text{(Division)} \end{array} \right\}$$

bilinear, symmetric, Lorentz Signature

Def<sup>n</sup>/ FIELDS are mappings from space-time  $M$  into a finite dimensional (here in this class) vector space  $V$ .

If  $\mathcal{F}(M, V)$  denotes the set of fields from  $M \rightarrow V$   
then  $\mathcal{F}(M, V) = \mathcal{F}$  is a vector space.

Now let  $\varphi \in \mathcal{F}$  and  $x \in M$ ,

$$\varphi(x) \in V$$

If  $\{e_a\}$  is a basis of  $V$  then

$$\varphi(x) = \sum_a \varphi^a(x) e_a$$

We assume  $\varphi^a \in C^\infty(M) \quad \forall a$ .

$$V = \mathbb{R}^n \text{ then } \varphi(x) = (\varphi^1(x), \varphi^2(x), \dots, \varphi^n(x))$$

$$e_a = (0, \dots, \underset{a^{\text{th}} \text{ spot.}}{1}, 0, \dots, 0)$$

(4)

Example One

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = g_{ab} \eta^{\mu\nu} (\partial_\mu \varphi^a)(\partial_\nu \varphi^b) + U(\|\varphi\|^2)$$

Assumptions

- M is minkowski space;  $(\mathbb{R}^4, \eta)$  where  $\|\eta^{\mu\nu}\| = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- $g$  is a metric on V; this is usually a positive definite metric.

Diversion

$$\begin{aligned} g(\varphi, \varphi) &= g(\varphi^a e_a, \varphi^b e_b) \text{ with } (V, M)^T \\ &= \varphi^a \varphi^b g(e_a, e_b) \text{ with } (V, M)^T \\ &= g_{ab} \varphi^a \varphi^b \end{aligned}$$

- $\varphi^a, \varphi^b \in C^\infty M$  and  $\partial_\mu \varphi^a, \partial_\nu \varphi^b \in C^\infty M$ ,  $g_{ab}, \eta^{\mu\nu} \in \mathbb{R}$   
So  $\mathcal{L}(\varphi, \partial_\mu \varphi) \in C^\infty M$ .
- $\|\varphi\|^2 = g_{ab} \varphi^a \varphi^b$  thus  $U: \mathbb{R}^+ \rightarrow \mathbb{R}^+$

EXAMPLE Two

From Q.M. : Schrod. Dirac... operation Procedure

(S)

$$E = \frac{1}{2m} P^2 + U(\mathbf{r})$$

$$E \rightarrow i\hbar \frac{\partial}{\partial t}$$

$$P \rightarrow -i\hbar \nabla$$

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$i\hbar \frac{\partial}{\partial t} = \frac{-1}{2m} \hbar^2 \nabla^2 + U$$

$$i\hbar \frac{\partial \varphi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \varphi + U \varphi$$

Problem with QM's is  $\varphi(t, x, y, z) \in \mathbb{C}$  so  $\varphi = u + iv$

where  $u, v : M \rightarrow \mathbb{R}$  so do we want to think of  $\mathbb{C}$  or  $\mathbb{R}^2$ ?

Well physics people tend to keep the Lagrangian complex but count real dimensions.  $\mathcal{L}(u, v, \partial_\mu u, \partial_\nu v)$

$$\mathcal{L}(\varphi, \varphi^\dagger, \partial_\mu \varphi, \partial_\nu \varphi^\dagger)$$

$$\varphi : M \rightarrow \mathbb{C}$$

$$(\varphi^\dagger = u - iv \text{ where } \varphi = u + iv)$$

The Lagrangian that yields Schrödinger boxed above is

$$\boxed{\mathcal{L}(\varphi, \varphi^\dagger, \partial_\mu \varphi, \partial_\nu \varphi^\dagger) = \frac{-\hbar^2}{2m} (\nabla \varphi^\dagger) \cdot (\nabla \varphi) + \frac{1}{2} i\hbar \varphi^\dagger (\partial_t \varphi) + \frac{1}{2} i\hbar \varphi \partial_t \varphi^\dagger - \varphi^\dagger U \varphi}$$

## Example 3

## Relativistic QM

$$\text{Relativistic Energy : } \frac{E^2}{c^2} - \vec{P}^2 = m^2 c^2$$

Operationalization of variables follows same prescription,

$$E \rightarrow i\hbar \frac{\partial}{\partial t}$$

$$\vec{P} \rightarrow -i\hbar \nabla$$

$$-\frac{\hbar^2}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + \hbar^2 \nabla^2 \varphi = m^2 c^2 \varphi$$

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = -\frac{m^2 c^2}{\hbar^2} \varphi \quad \text{Klein-Gordan Eq. 1}$$

## Example 4

## Maxwell's Lagrangian

## Example 5 Yang-Mills Lagrangian

- Basically all matter fermions, basically Dirac Eq. 5 governs at base level spin  $\frac{1}{2}$  particles.
- Force Particles are integer spin bosons: photons,  $W \pm Z$ , nuclear mesons etc. all of these are representable by a Yang-Mills field.

Def<sup>n</sup>/ YANG-MILLS (field) is a (positive) Lie Algebra valued one-form on M

$$A = A_\mu dx^\mu$$

$$A\left(\frac{\partial}{\partial x^\nu}\right) = A_\mu dx^\mu \left(\frac{\partial}{\partial x^\nu}\right) = A_\nu \in \mathfrak{g} = G'$$

$G$  a lie group

$$\mathfrak{g} = \mathfrak{gl}(n)$$

$\{e_a\}$  basis of  $\mathfrak{g}$

$$A = (A_\mu^a e_a) dx^\mu$$

Usually folks write

$$A = A_\mu^a (dx^\mu \otimes e_a)$$

$$A(x) = A_\mu^a (dx^\mu \otimes e_a)(x) = A_\mu^a dx^\mu(x) e_a$$

$A$  is a Lie-Algebra valued 1-form on a manifold  $M$ .  
 $M = \text{Minkowski}^4$ . Also  $\mathfrak{g}$  Lie Algebra of group  $G$ .

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z); \text{ Global Chart on } M.$$

Now we can take  $A_\nu$  out of  $\mathfrak{g}$   $\leftarrow$  restricted Lie A/g. details to follow.

$$A = A_\nu dx^\nu; A_\nu \in \mathfrak{g}$$

$$A\left(\frac{\partial}{\partial x^\nu}\right) = A_\nu \in \mathfrak{g}$$

Suppose that  $\langle e_a \rangle = \mathfrak{g}$  then  $A_\nu = A_\nu^a e_a$ ,  $A_\nu^a \in C^\infty M$

With some reservation we write

$$A = A_\nu^a (dx^\nu \otimes e_a)$$

Then consider,

$$F_A = F(A) = dA + \frac{1}{2}[A, A]$$

$$[A, A](\mathbf{x}, \mathbf{y}) = [A(\mathbf{x}), A(\mathbf{y})]_{\mathfrak{g}}$$

↑  
not a Lie Algebraic  
Bracket, but we are  
defining it in terms  
of  $[, ]$

$$[A, A]\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = [A\left(\frac{\partial}{\partial x^\mu}\right), A\left(\frac{\partial}{\partial x^\nu}\right)] = [A_\mu, A_\nu]$$

(?) - Mtheory ... ?

$$A = \sum A_\nu dx^\nu$$

$$\begin{aligned} A\left(\frac{\partial}{\partial x^\nu}\right) &= \sum A_\nu dx^\nu \left(\frac{\partial}{\partial x^\nu}\right) \\ &= A_\nu \end{aligned}$$

(2)

MA 797-0

Note:

$$A_\mu = A_\mu^a e_a$$

$$\frac{\partial A_\mu}{\partial x^\nu} = \frac{\partial A_\mu^a}{\partial x^\nu} e_a$$

$$\begin{aligned}
 dA &= \sum_\mu dA_\mu \wedge dx^\mu \\
 &= \sum_\mu \sum_\nu \frac{\partial A_\mu}{\partial x^\nu} (dx^\nu \wedge dx^\mu) \\
 &= \frac{1}{2} \sum_{\mu, \nu} \left( \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) dx^\nu \wedge dx^\mu \\
 &= \frac{1}{2} (A_{\mu, \nu} - A_{\nu, \mu}) (dx^\nu \wedge dx^\mu) \\
 &= \frac{1}{2} (A_{\mu, \nu}^a - A_{\nu, \mu}^a) (dx^\nu \wedge dx^\mu) \otimes e_a
 \end{aligned}$$

If  $dA$  acts on pair of vectors we contract them with  $dx^\nu$  and  $dx^\mu$ . In practice he usually just looks at

$$\begin{aligned}
 dA(\partial_\rho, \partial_\lambda) &= \frac{1}{2} (A_{\mu, \nu}^a - A_{\nu, \mu}^a) (dx^\nu \wedge dx^\mu) (\partial_\rho, \partial_\lambda) e_a \\
 &= \frac{1}{2} (A_{\mu, \nu}^a - A_{\nu, \mu}^a) \frac{1}{2} (dx^\nu \otimes dx^\mu) (\partial_\rho, \partial_\lambda) e_a \\
 &= \frac{1}{4} (A_{\mu, \nu}^a - A_{\nu, \mu}^a) (\delta_\nu^\rho \delta_\lambda^\mu - \delta_\nu^\lambda \delta_\mu^\rho) e_a \\
 &= \left\{ \frac{1}{4} (A_{\lambda, \rho}^a - A_{\rho, \lambda}^a) + \frac{1}{4} (A_{\rho, \lambda}^a - A_{\lambda, \rho}^a) \right\} e_a
 \end{aligned}$$

$$dA(\partial_\rho, \partial_\lambda) = \frac{1}{2} (A_{\lambda, \rho}^a - A_{\rho, \lambda}^a) e_a$$

a sum over dimension of Lie Algebra

So we can express the field strength  $F_{\mu\nu}(A)$  as,

$$F_{\mu\nu}(A) = dA(\partial_\mu, \partial_\nu) + \frac{1}{2} [A, A](\partial_\mu, \partial_\nu)$$

$$F_{\mu\nu}(A) = \frac{1}{2} (A_{\nu, \mu}^a - A_{\mu, \nu}^a) e_a + \frac{1}{2} [A_\mu^a e_a, A_\nu^b e_b]$$

$$F_{\mu\nu}(A) = \frac{1}{2} (A_{\nu, \mu}^a - A_{\mu, \nu}^a) e_a + \frac{1}{2} A_\mu^a A_\nu^b [e_a, e_b]$$

(3)

Def<sup>n</sup> / Structure Constants of Lie Algebra.

$$[e_a, e_b] = \sum_{c=1}^{\dim(\mathfrak{g})} f_{ab}^c e_c$$

Where  $f_{ab}^c = -f_{ba}^c$  for what that's worth.

Continuing on our strange diversion; Jacobi Identity

$$[e_a, [e_b, e_c]] + [e_b, [e_c, e_a]] + [e_c, [e_a, e_b]] = 0$$

We can look up how this looks in structure constants.  
Clearly it wouldn't take long to derive. So we  
can express  $dA$  with structure constants,

$$\cancel{dA} F_{\mu\nu}(A) = \left[ \frac{1}{2} (A_{\nu,\mu}^c - A_{\mu,\nu}^c) + \frac{1}{2} f_{ab}^c A_\mu^a A_\nu^b \right] e_c$$

Physics People  $\rightarrow$   $F_{\mu\nu}^c(A) = \frac{1}{2} [\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + f_{ab}^c A_\mu^a A_\nu^b]$

So he didn't plan all this (oops we did part of the exam)

———— //

- We assume  $\mathfrak{g}$  is a finite direct sum of commuting Lie Algebras each of which is either the Lie Algebra of compact simple Lie group or is the Lie Algebra of  $U(1)$ . In this case on each factor one may define a positive definite metric  $g$  by

$$g_{ab} = -\text{Trace}(e_a e_b)$$

Where  $\{e_a\}$  is a basis of  $\mathfrak{g}$ . Trace is symmetric  $\therefore g_{ab} = g_{ba}$

Now if  $u = u^a e_a$ ,

$$\begin{aligned} g(u, u) &= g(u^a e_a, u^b e_b) \\ &= u^a u^b g(e_a, e_b) \stackrel{?}{=} 0 \\ &= -\text{Tr}(e_a e_b) u^a u^b \\ &= \frac{-\text{Tr}(uu)}{\text{Tr}(uu)} \end{aligned}$$

$\bar{M}^+ = -M$

$$\text{Tr}(M^2) = \sum_a (M^2)_a^a = \sum_{a,b} M_b^a M_a^b = - \sum_{a,b} M_b^a \overline{M}_b^a \therefore g(u, u) = -(-\sum_b M_b^a M_a^b)$$

Q!

9)  $A^2 + B^2 = C^2$  (Pythagorean theorem)

$$(A^2 + B^2)^2 = (A^2 + B^2)(A^2 + B^2)$$

and  $(A^2 + B^2)^2 = A^4 + 2A^2B^2 + B^4$  (square of sum)

so  $A^4 + B^4 + 2A^2B^2 = A^4 + B^4 + C^4$  (Pythagorean theorem)

so  $2A^2B^2 = C^4 - A^4 - B^4$  (canceling terms)

so  $A^2B^2 = \frac{C^4 - A^4 - B^4}{2}$  (divide by 2)

$\therefore A^2B^2 = \frac{C^4 - A^4 - B^4}{2}$

$$(A^2B^2)^{\frac{1}{2}} = \sqrt{\frac{C^4 - A^4 - B^4}{2}}$$

so  $\sqrt{A^2B^2} = \sqrt{\frac{C^4 - A^4 - B^4}{2}}$  (square root both sides)

$$\therefore \sqrt{AB} = \sqrt{\frac{C^4 - A^4 - B^4}{2}}$$

$\therefore \sqrt{AB} = \sqrt{\frac{C^4 - A^4 - B^4}{2}}$

- ① Oliver Problem due this Friday
- ② Problem 4 by Friday After
- ③ FINAL EXAM ? ← Mad Calculation Beware

FINAL EXAM CLARIFICATION

$G$  matrix Lie Group where  $G \subseteq \mathrm{SL}(n)$

$G' \equiv \mathfrak{g} \equiv$  Lie Algebra of  $G \subseteq \mathfrak{u}(n)$

$\mathfrak{u}(n)$  is Lie Algebra of  $U(n) = \{A \mid A^{\text{ct}} A = I\}$

$\mathfrak{u}(n) = \{A \in \mathrm{gl}(n, \mathbb{C}) \mid A^{\text{ct}} = -A\}$

We define  $g$  a metric on  $\mathfrak{g}$  by ( $g$  is positive definite)

$g(e_a, e_b) = -\text{Trace}(e_a e_b)$  where  $\{e_a\}$  is basis of  $\mathfrak{g}$

$u_1, u_2 \in \mathfrak{g} \Rightarrow u_1 = u_1^a e_a$  and  $u_2 = u_2^b e_b$  then using  $g(e_a, e_b) = g_{ab} \in \mathbb{R}$

$$g(u_1, u_2) = u_1^a u_2^b g(e_a, e_b) = u_1^a u_2^b g_{ab}$$

Now the basis may be chosen such that (Weinberg QTF's)

$$(1.) \quad g_{ab} = \delta_{ab}$$

(2.) If  $[e_a, e_b] = f_{ab}^c e_c$  then ~~fabc~~  $f_{cab} = g_{dc} f_{ab}^d$  is

skew symmetric ( $f_{abc} = -f_{acb} = -f_{bac}$ )

(this at least works for subalgebra's of  $U(n)$ )

technically we work with compact semi-simple such groups can be contained in  $U(n)$

$$U(n) = \{A^{\text{ct}} = A^{-1}\}$$

$$U(n) = \{A \in \mathrm{SL}(n) \mid A^{\text{ct}} A = I\}$$

(2)

$A^a$  ← where in the Lie Algebra  
 $A_\mu$  ← which component in Space-time

$$A = A_\mu^a (dx^\mu \otimes e_a) \in T^*M \otimes \mathfrak{g}$$

remember this just goes for ride

## THE LAGRANGIAN

$$\mathcal{L}(A_\mu^a, \partial_\nu A_\mu^a) = \frac{1}{4} g_{ab} \tilde{\eta}(F^a(A), F^b(A))$$

real valued  
two-form

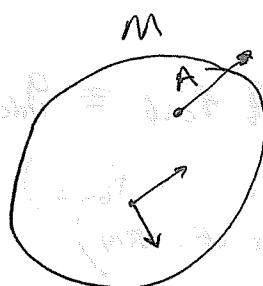
Field Strength  $\rightarrow F(A) = dA + \frac{1}{2} [A, A]$

$a^{th}$  coordinate  $\rightarrow F^a(A) = \frac{1}{2} \{ \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c \} (dx^\mu \wedge dx^\nu)$

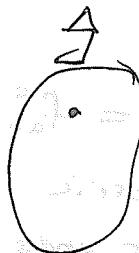
in Lie Algebra  
of the field  
strength.

$$\mathcal{L}\{A_\mu^a, \partial_\nu A_\mu^a\} = \frac{1}{4} \eta^{\mu\nu} \eta^{\rho\kappa} g(F_{\mu\rho}^a e_a, F_{\nu\kappa}^b e_b)$$

$C^\infty$  (Minkowski)



one inner-product  
on minkowski can be  
beefed up to metric  
on two forms



inner product  
of elements  
in Lie Algebra

$$\mathcal{L}(A_\mu^a, \partial_\nu A_\mu^a) = \frac{1}{4} g_{ab} \eta^{\mu\nu} \eta^{\rho\kappa} F_{\mu\rho}^a F_{\nu\kappa}^b$$

use this one  $\rightarrow = \frac{1}{4} \delta_{ab} \eta^{\mu\nu} \eta^{\rho\kappa} F_{\mu\rho}^a F_{\nu\kappa}^b$  ← choose happy physics basis  
for all our calculations for the sake of ease.

$$= -\frac{1}{4} \eta^{\mu\nu} \eta^{\rho\kappa} \text{Trace}(F_{\mu\rho} F_{\nu\kappa}) \quad \text{where } F_{\mu\nu} = F_{\mu\nu}^a e_a$$

matrix

need this for showing  
Lagrangian invariant under

(3)

① Find Euler-Lagrange eq<sup>n</sup>'s

② Show Lagrange Eq<sup>n</sup> invariant under group. We have a group that acts on a sol<sup>n</sup> to lag. Eq<sup>n</sup> and yields another sol<sup>n</sup> - this generalizes the strategy of Galois.

## Euler-LAGRANGE EQUATIONS

What are Euler-Lagrange Equations?

Why are they Import?

Where did they come from?

\* They came from Classical Mechanics

$$K = \frac{1}{2} m (\vec{v} \cdot \vec{v}) \quad \text{Kinetic Energy}$$

$\vec{v}$  = velocity vector

$m$  = mass

If  $U$  is the potential of a conservative force field in which the object moves the total energy is  $K+U = E$

$$E(x^1, x^2, x^3, v^1, v^2, v^3) = \sum_{i=1}^3 \frac{m}{2} v_i^2 + U(x^1, x^2, x^3)$$

$$E: TM \rightarrow \mathbb{R}$$

If we change  $v^i \rightarrow p^i$  covectors then we have Hamiltonian.

$$\mathcal{L}(x^1, x^2, x^3, v^1, v^2, v^3) = \frac{m}{2} \sum_{i=1}^3 (p_i)^2 - U(x^1, x^2, x^3)$$

It is easy to show that the following set of eq<sup>n</sup>'s is equivalent to Newtons Eq<sup>n</sup>'s;

$$\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial v^i} \right) = 0$$

These eq<sup>n</sup>'s are invariant under coordinate change unlike Newtons Laws; for example if we go to polar coordinates in Newton's formalism then we need to introduce fictitious forces etc...

(4)

If we change coordinates from  $(x^1, x^2, x^3) \rightarrow (g^1, g^2, g^3)$

$$K = \frac{m}{2} g_{ij} \dot{g}^i \dot{g}^j$$

Where  $g$  is positive def<sup>+</sup> metric on  $M$ . While the potential becomes

$$U = U(g^1, g^2, g^3)$$

And then the Euler Lag. Eq <sup>$\pm$</sup> s become

$$\frac{\partial L}{\partial \dot{q}^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0 \quad t \rightarrow g(t)$$

This is the particle-formalism

The generalization to fields is then

$$\frac{\partial L}{\partial \dot{\varphi}^a} - \frac{\partial}{\partial x^\mu} \left[ \frac{\partial L}{\partial (\partial_x \varphi^a)} \right] = 0 \quad (\text{Poor Notation})$$

$\varphi : M \rightarrow V \leftarrow$  finite dimensional Vect Space  
with basis  $e_a$

$$\varphi(x) = \sum \varphi^a(x) e_a$$

Technically if we take  $M = \mathbb{R}^3$ ,  $V = \mathbb{R}^3$  and  $g = \varphi : \mathbb{R} \rightarrow \mathbb{R}^3$

$\dot{g}(t) = (g^1(t), g^2(t), g^3(t)) = \sum g^i(t) e_i$  this reduces to the particle lagrangian.

$$L = \frac{1}{2} m \dot{g}^i \dot{g}^i$$

(5)

Example

$$\mathcal{L} = C_{ab}^\mu \varphi^a (\partial_\nu \varphi^b)$$

$$\varphi : \mathbb{R}^3 \rightarrow V$$

$$\varphi(x^i) \in V \ni \varphi(x) = \varphi(x^1, x^2, x^3)$$

$$\varphi(x) = \sum_{a=1}^n \varphi^a(x) e_a$$

$$\mathcal{L}(\varphi^1, \varphi^2, \dots, \varphi^n, \partial_1 \varphi^1, \partial_2 \varphi^1, \dots, \partial_3 \varphi^1, \partial_1 \varphi^2, \dots, \partial_3 \varphi^n) \leftarrow \text{lots of variables}$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial \varphi^d} = C_{ab}^\mu \delta_d^a (\partial_\nu \varphi^b) = C_{db}^\mu (\partial_\nu \varphi^b)$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\nu \varphi^d)} = C_{ab}^\mu \varphi^a \delta_d^b \delta_\nu^\mu = C_{ad}^\nu \varphi^a$$

So the Euler Lagrange Eq's are

$$\frac{\partial \mathcal{L}}{\partial \varphi^d} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu \varphi^d)} \right) = 0$$

$$C_{db}^\mu \partial_\nu \varphi^b - \partial_\nu (C_{ad}^\nu \varphi^a) = 0$$

$$C_{db}^\mu \partial_\nu \varphi^b - C_{ad}^\nu \partial_\nu \varphi^a = 0$$

$$C_{da}^\mu \partial_\nu \varphi^a - C_{ad}^\mu \partial_\nu \varphi^a = 0$$

$$\cancel{C_{da}^\mu} \left( \cancel{\partial_\nu \varphi^a} - \cancel{\partial_\nu \varphi^a} \right) = 0$$

$$(C_{da}^\mu - C_{ad}^\mu)(\partial_\nu \varphi^a) = 0$$

$$\bar{C}_{da}^\mu \partial_\nu \varphi^a = 0$$

$$\left( \sum_r \bar{C}_{da}^\mu \frac{\partial}{\partial x^r} \right) \varphi^a = 0$$

Linear PDE  
with constant  
coefficients

(

the  $\lambda$  and  $\mu$  components of the vector field  $\vec{v}$ .

$$\partial^{\alpha} \vec{v} = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$(\partial^{\alpha} v_x) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \sqrt{g} v_x \right) - \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \sqrt{g} v_y \right) - \frac{1}{\sqrt{g}} \frac{\partial}{\partial z} \left( \sqrt{g} v_z \right)$$

similarly for  $v_y$  and  $v_z$ .

$$\nabla_{\vec{v}} \vec{v} = \partial^{\alpha} \vec{v} = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k} \quad (\text{Eq. 6})$$

$$\nabla_{\vec{v}} \vec{v} = \partial^{\alpha} \vec{v} = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k} \quad (\text{Eq. 6})$$

where  $\partial^{\alpha}$  is the covariant derivative operator defined by the metric tensor  $g_{\mu\nu}$ .

$$(\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$v = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$v = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$v = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$v = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$v = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$v = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$v = (\partial^{\alpha} v_x) \hat{i} + (\partial^{\alpha} v_y) \hat{j} + (\partial^{\alpha} v_z) \hat{k}$$

$$\text{Let } d\lambda^i = \lambda_k^i dx^k \quad \text{then } ?$$

$$d(\lambda^i) = d\lambda^i = \lambda_k^i dx^k$$

$$d\lambda^i = (\lambda_1^i, \lambda_2^i, \dots)$$

$$\text{Transform } d\lambda^i = \frac{\partial \lambda^i}{\partial x^k} dx^k$$

~~May 6, 1996~~

Due dates: May 6, 1996 - (P.S., S, M) - Friday

Friday - Oliver Problem

Friday May 9<sup>th</sup> - All Hawk.

Thursday Noon May 10<sup>th</sup> - Final Exam

Teacher Evaluation April 30<sup>th</sup> Monday

~~May 6, 1996~~

~~May 6, 1996~~

(2)

## LAGRANGIANS AND COMPANY

$\varphi: M \rightarrow V$  where  $M = \mathbb{R}^4$  (basis  $\{e_a\}$ )

$$\varphi(x) = \sum_{a=1}^N \varphi^a(x) e_a$$

$$L(\varphi^a, \partial_\mu \varphi^b) = c_{ab}^\mu \varphi^a (\partial_\mu \varphi^b) \quad \varphi^a \in C^\infty(M)$$

$L$  the lagrangian is a function of field components and field derivatives.  $0 \leq \mu \leq 3$

$$E_L : (c_{db}^\mu - c_{bd}^\mu) \partial_\mu \varphi^b = 0 \quad 1 \leq d \leq N$$

$$E_L(\varphi, \partial_\mu \varphi) = \frac{\partial L}{\partial \varphi^d} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \varphi^d)} \right)$$

But  $E_L$  appears to infer differentiation of functions.  $\Rightarrow$  function space  $\Rightarrow$  which one (?) thus we begin to introduce the Jet Bundle to avoid the question until it's phrased nicer.

$(x^\mu)$  coordinates on  $M$

$(u^a)$  coordinates on  $V$

$u_\nu^b \leftarrow$  represent the derivatives but are not the derivatives at present.

$$v = \sum_b v^b e_b$$

$$u^a(v) = v^a$$

$$\begin{array}{ccc} M \times V \times \mathbb{R}^{DN} & \approx & \mathbb{R}^4 \times \mathbb{R}^N \times \mathbb{R}^{DN} \\ \downarrow & \downarrow & \downarrow \\ x^\mu & u^a & u_\nu^a \end{array}$$

③

4/24/01

MA 793

$$\mathcal{L}(x^\mu, u^a, u_\nu^a) = C_{ab}^\mu u^a u_\nu^b$$

$$\Sigma_d \mathcal{L}(x^\mu, u^a, u_\nu^a) = \frac{\partial \mathcal{L}}{\partial u^a}(x^\mu, u^a, u_\nu^a) - \frac{\partial}{\partial x^\nu} \left( \frac{\partial \mathcal{L}}{\partial u_\nu^a} \right)$$

$$\Sigma_d \mathcal{L} : M \times V \times \mathbb{R}^{NO}$$

$$\frac{\partial \mathcal{L}}{\partial u^a} = C_{db}^\mu u_\nu^b$$

$$\frac{\partial \mathcal{L}}{\partial u_\nu^a} = C_{ad}^\nu u^a$$

$$\Sigma_d \mathcal{L} = C_{db}^\mu u_\nu^b - \frac{\partial}{\partial x^\nu} (C_{ad}^\nu u^a)$$

How do we need to def<sup>n</sup> these characters to work properly?

$$\frac{\partial}{\partial x^\mu} (C_{ad}^\nu \varphi^a) = C_{ad}^\nu \partial_\mu \varphi^a$$

What is  $\frac{\partial}{\partial x^\mu}$ ? Suppose we have any function

$$f : M \times V \times \mathbb{R}^{NO} \rightarrow \mathbb{R} ; f(x^\mu, u^a, u_\nu^b) \in \mathbb{R}$$

$$f(x, \varphi^a(x), \partial_\nu \varphi^b(x)) = \frac{\partial f}{\partial x^\mu} + \frac{\partial f}{\partial u^a} \frac{\partial \varphi^a}{\partial x^\mu} + \frac{\partial f}{\partial u_\nu^b} \frac{\partial^2 \varphi^b}{\partial x^\mu \partial \nu}$$

$$= \left( \frac{\partial}{\partial x^\mu} + u_\nu^a \frac{\partial}{\partial u^a} + u_\nu^b \frac{\partial}{\partial u_\nu^b} \right) (f)$$

This tells us the  $\Sigma_d$  wasn't quite right, enough derivatives have we not to the 2<sup>nd</sup> Jet Bundle we must go and the total derivative define we next.

(4)

Defn / THE TOTAL DERIVATIVE

$$\partial_r^T = \frac{\partial}{\partial x^n} + u_\mu^a \frac{\partial}{\partial u^a} + u_{\mu\nu}^a \frac{\partial}{\partial u_\nu^a}$$

Kowach  
Hilbert  
PDE's

$\partial_r^T$  is a vector field on  $J^2 E$

$$E = M \times V \quad (x^n, u^a)$$

$$J^1 E = M \times V \times \mathbb{R}^{ND} \quad (x^\mu, u^a, u_\nu^b)$$

$$J^2 E = M \times V \times \mathbb{R}^{NO} \times \mathbb{R}^m \quad (x^\mu, u^a, u_\nu^b, u_{\mu\nu}^c)$$

$J^1 E$  is the 1<sup>st</sup> Jet Bundle.

$J^2 E$  is the 2<sup>nd</sup> Jet Bundle

$J^2 E$  is where we need to define  $d$  on.

$$L(x^\mu, u^a, u_\nu^b, u_{\mu\nu}^c) \quad \text{sometimes } L \text{ really needs all these but not always}$$

$$d_L L = \frac{\partial L}{\partial u^a} - \partial_r^T \left( \frac{\partial L}{\partial u^a} \right)$$

$$L = C_{ab}^\mu u^a u_\nu^b$$

$$\begin{aligned} d_L L &= C_{db}^\mu u_\nu^b - \partial_\nu (C_{ad}^\nu u^a) \\ &= C_{db}^\mu u_\nu^b - u_\nu^a C_{ad}^\nu \end{aligned}$$

$$\begin{aligned} \partial_\nu^T f_{yd} &= (\partial_\nu^b \frac{\partial}{\partial u^b})(C_{ad}^\nu u^a) \\ &= (u_\nu^b \frac{\partial}{\partial u^b})(C_{ad}^\nu u^a) \\ &= u_\nu^b C_{ad}^\nu \delta_b^a \\ &= u_\nu^a C_{ad}^\nu \end{aligned}$$

$$= C_{da}^\mu u_\nu^a - u_\nu^a C_{ad}^\mu$$

$$= (C_{da}^\mu - C_{ad}^\mu) u_\nu^a$$

$$(5) \quad \text{such that } \sum_d L_d = (\bar{C}_{da}^{\mu} - \bar{C}_{ad}^{\mu}) u_{\mu}^a$$

This def<sup>ns</sup> a level surface in  $J^2 E$ . Specifically,

$$\Sigma^{(2)} = \left\{ (x^\mu, u^a, u_r^a, u_{\mu\nu}^c) \in J^2 E \mid \sum_d L_d (x^\mu, u^a, u_r^a, u_{\mu\nu}^c) = 0 \right\}$$

Now  $\Sigma^{(2)}$  is regular sub-manifold of  $J^2 E$  provided  $\bar{C}_{da}^{\mu} u_{\mu}^a = 0$  has maximal rank.

$$\Sigma^{(2)} = \left\{ (x^\mu, u^a, u_r^a, u_{\mu\nu}^c) \in J^2 E \mid \bar{C}_{da}^{\mu} u_{\mu}^a = 0 \right\}$$

A sol<sup>n</sup> of the Euler Lagrange Eq<sup>n</sup> is a mapping  $\tilde{\varphi} : M \rightarrow E = M \times V$  such that  $\sum_d L_d \circ j^2 \tilde{\varphi} = 0$   
where  $\tilde{\varphi}(x) = (x, \varphi(x))$ ,  $\varphi : M \rightarrow V$ . A map such as  $\tilde{\varphi}$  is a section (it is a trivial section because  $M \times V$  is a trivial bundle). Define

$$j^2 \tilde{\varphi} : M \longrightarrow J^2 E$$

$$(j^2 \tilde{\varphi})(P) = (x^\mu(P), u^a(\varphi(P)), u_r^a(\varphi(P)))$$

$$u^a(v) = v^a$$

$$u^a(\varphi(P)) = \varphi^a(P)$$

$$(u_r^a \varphi)(P) = \frac{\partial}{\partial x^r} (\varphi^a(P))$$

$$j^2 \tilde{\varphi} : M \rightarrow J^2 E$$

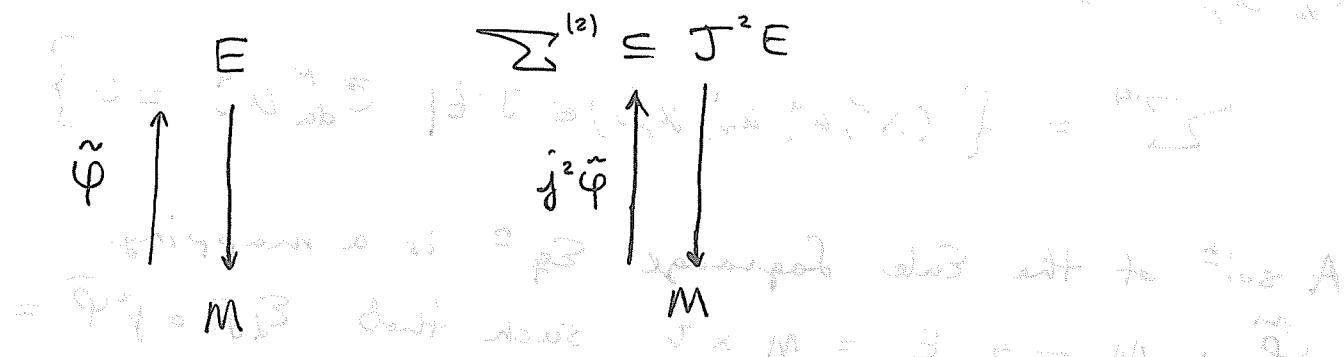
$$\sum_d L_d : J^2 E \rightarrow \mathbb{R}$$

⑥

We take Lagrange Eq<sup>h</sup> and try to break it into two pieces. Eq<sup>h</sup> identified with surface then sol<sup>h</sup> identified as mapping onto surface I think.

$$\sum_d L \circ j^2 \tilde{\varphi} = 0$$

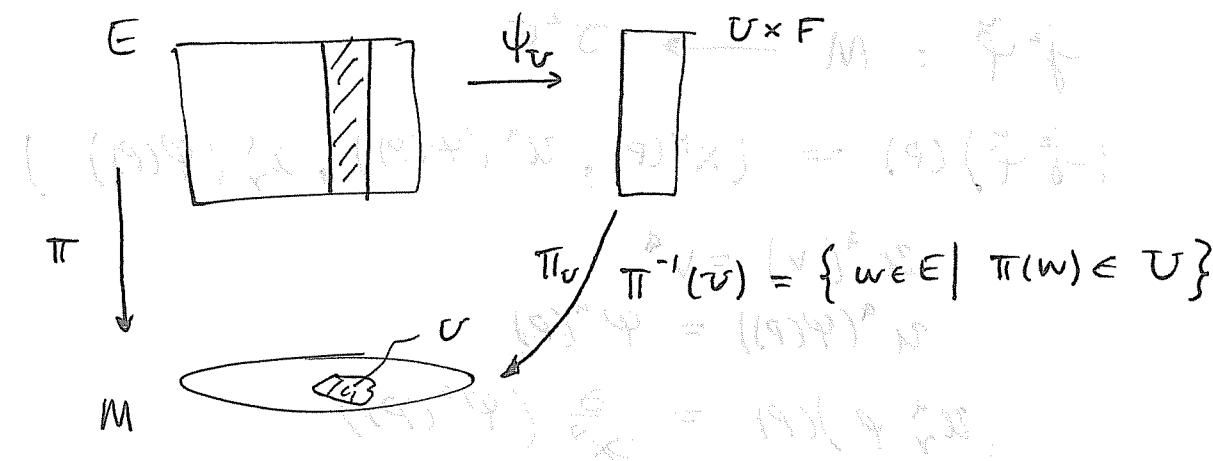
$$(j^2 \tilde{\varphi})(P) \in \sum^{(2)}$$



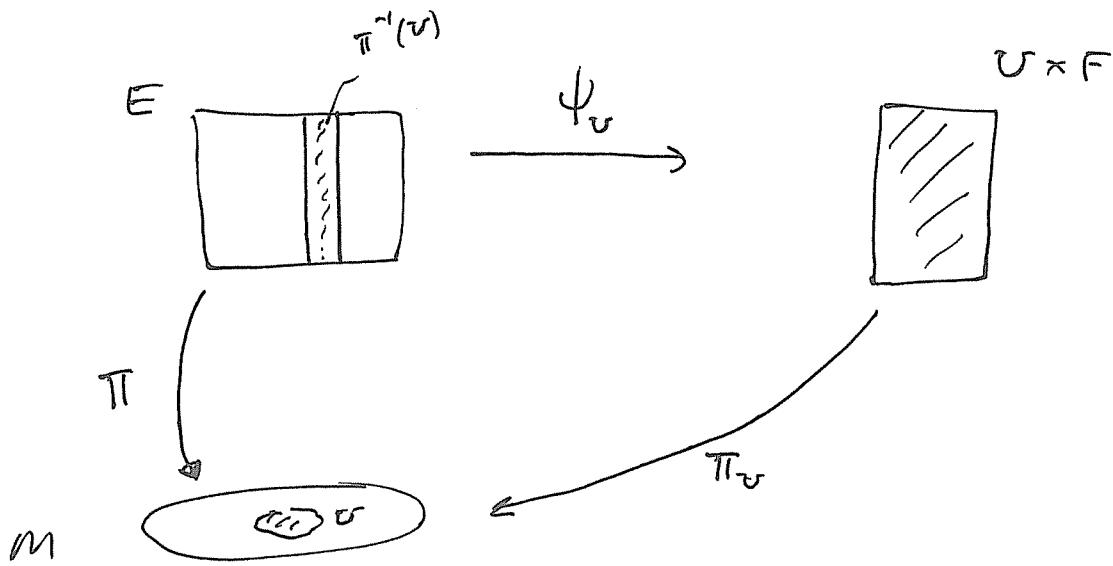
$\varphi = \varphi^1 \cdot M^3$  don't want  $M \times M \times S \subset M \times M$

How to generalize this stuff

Assume that  $E \xrightarrow{\pi} M$  is a fibre bundle



The fibre is usually a group or a vector space.



Examples

$$E = TM$$

$$\pi : TM \rightarrow M$$

$$\pi(p, v) = p$$

$(U, x)$  is a chart on  $M$

$$\psi_v(x, v) = (x(p), dx^\mu(v) e_\mu) = (x(p), dx(v))$$

$$\psi_v : \pi^{-1}(U) \rightarrow x(U) \times \mathbb{R}^{\dim(M)} \rightarrow (p, dx(v)) \in U \times \mathbb{R}^m$$

$$E = T^*M$$

$$\tau(p, \alpha) = p$$

$$\psi_v(p, \alpha) = (p, \alpha(\frac{\partial}{\partial x^\mu}) e_\mu)$$

$M \rightarrow \pi^+$

$M \rightarrow M T \rightarrow \pi$

$\pi \rightarrow (\pi\pi)T$

$\pi \rightarrow \pi \pi \rightarrow \pi(\pi\pi)$

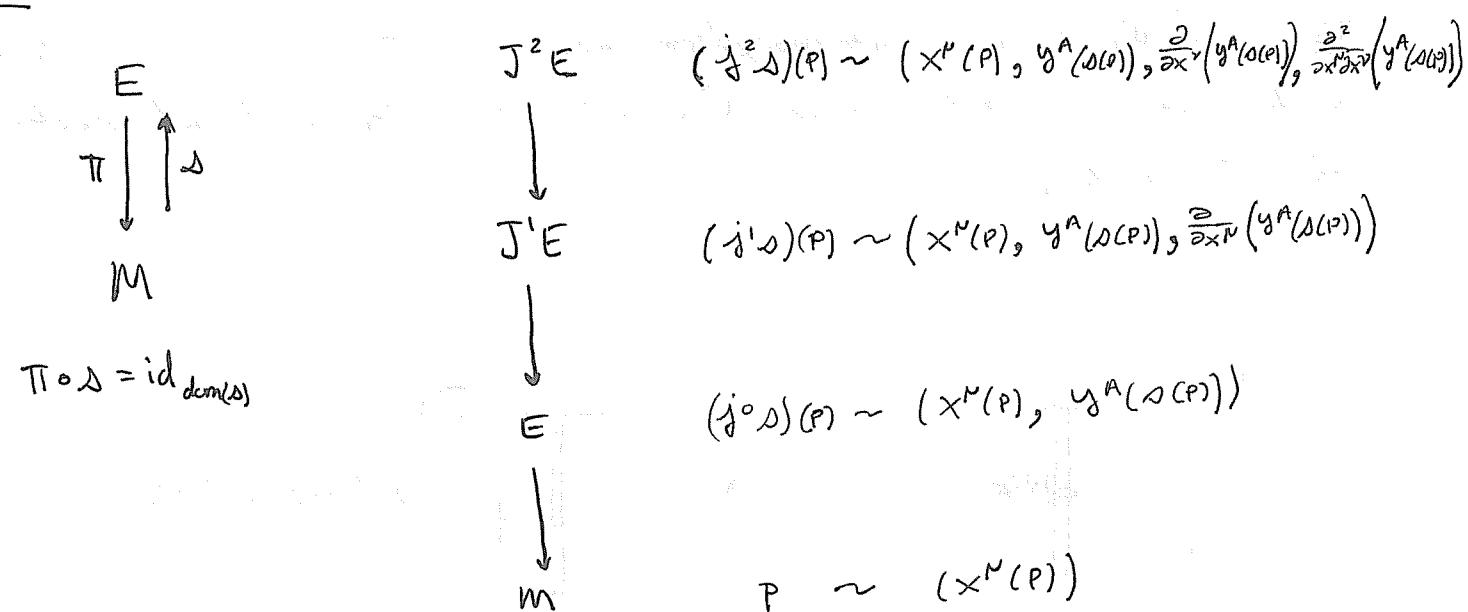
first link = last link  $\Rightarrow$   $(\pi, \pi)_L$

start is  $(\pi, \pi)_L$  and end is  $(\pi, \pi)_R$   $\Rightarrow$   $(\pi, \pi)_R$

$M T \rightarrow \pi$

$\pi \rightarrow (\pi\pi)T$

$\pi \rightarrow (\pi\pi\pi)T$



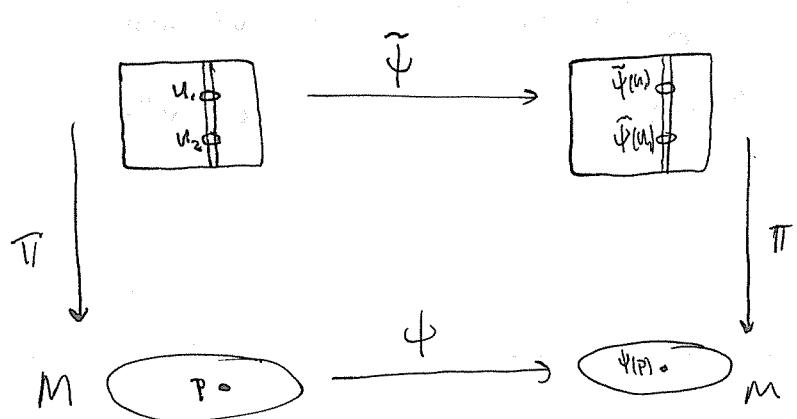
- A symmetry is a diffeomorphism we use to change coordinates with

(Defn) To say that  $(\tilde{\psi}, \psi)$  is an automorphism of the Bundle  $E \rightarrow M$  means that  $\tilde{\psi}$  and  $\psi$  are diffeomorphisms such that

$$\tilde{\psi} : E \rightarrow E \quad \text{and} \quad \psi : M \rightarrow M \quad \text{and}$$

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\psi}} & E \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\psi} & M \end{array}$$

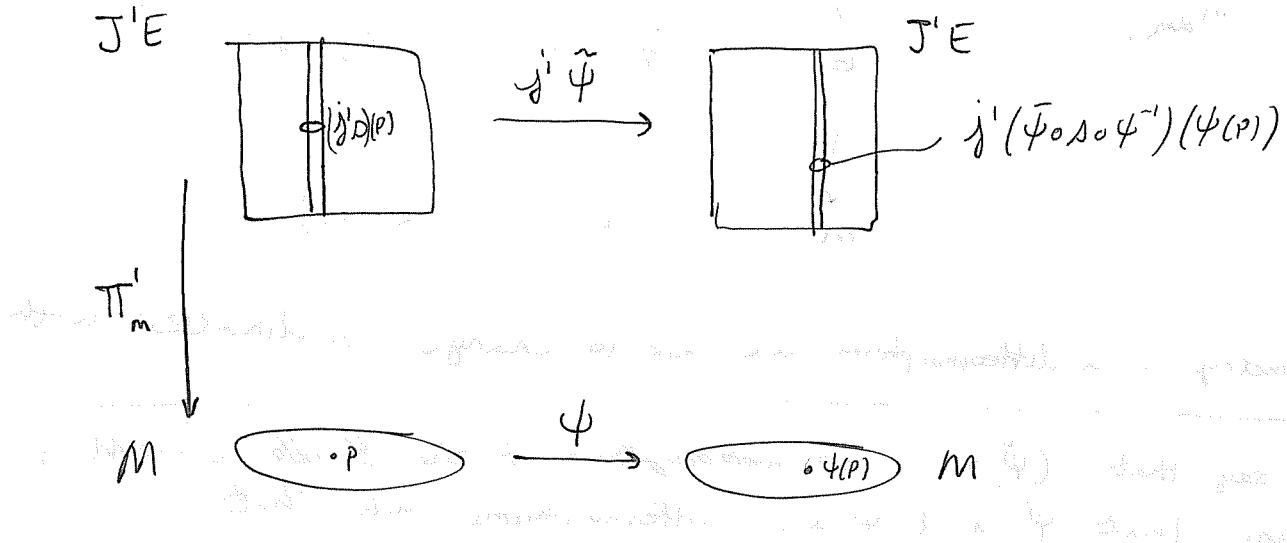
is a commutative diagram,  $\pi \circ \tilde{\psi} = \psi \circ \pi$ .



If  $u_1, u_2 \in \pi^{-1}(p) \equiv E_p$ , then he claims  $\tilde{\psi}(u_1), \tilde{\psi}(u_2) \in E_{\tilde{\psi}(p)} = \pi^{-1}(\psi(p))$ , it sends fibres to fibres.

If  $(\tilde{\psi}, \psi)$  is an automorphism and  $s: V \rightarrow E$  is a local section of  $E \rightarrow M$  then  $\tilde{\psi} \circ s \circ \psi^{-1}$  is also a local section of  $E \rightarrow M$ .

$\frac{1}{2}$



$(j^*\tilde{\psi}, \psi)$  is an automorphism of  $J'E \rightarrow M$ .

$\frac{1}{2}$

$$\mathcal{L}: J'E \rightarrow \mathbb{R}$$

$$\mathcal{L}((j^*s)(x)) = \mathcal{L}(x^\mu(p), y^A(s(p)), \partial_\mu(y^A(s(p))))$$

$$\mathcal{L}(\tilde{\psi}^*(\tilde{\psi})(j^*s(p))) = \mathcal{L}(\bar{x}^\mu(p), \bar{y}^A(s(p)), \bar{\partial}_\mu(\bar{y}^A(s(p))))$$

$$(\bar{x}^\mu(p), \bar{y}^A(s(p)), \bar{\partial}_\mu(\bar{y}^A(s(p)))) = (j^*\tilde{\psi}(x^\mu(p), y^A(p), \partial_\mu(y^A(s(p))))$$

- ① In physics when is  $\mathcal{L}$  invariant under certain types of transformations (automorphisms)
- ② Action invariant
- ③ Eg's of motions invariant this is the order of importance with the symmetries.

(3)

If  $\tilde{\gamma}$  is a vector field on  $E$  such that its flow  $\{\tilde{\phi}_t\}$  is a curve of automorphisms of  $E$ , then one can define a vector field on  $J^1 E$  by (Prolongation)

$$P_n'(\tilde{\gamma})(w) = \left. \frac{d}{dt} (\tilde{j}'(\tilde{\phi}_t))(w) \right|_{t=0}$$

The prolongation is an infinitesimal lift while  $\tilde{\gamma}$  is said to be an infinitesimal automorphism. Let  $E$  be acted on by a group who sends fibre  $\rightarrow$  fibre then we are off and running.

### THEOREM

Let  $E \rightarrow M$  be a fibre bundle and  $L : J^1 E \rightarrow \mathbb{R}$  a smooth map. Let  $(\tilde{\psi}, \psi)$  be an automorphism then,

①  $L$  is invariant under  $(\tilde{\psi}, \psi)$  iff  $L(\tilde{j}'(\tilde{\psi} \circ s)(x)) = L((js)(x))$

A section  $s$   
of  $E \rightarrow M$   
 $\forall x$

② The Automorphism  $(\tilde{\psi}, \psi)$  is a variational symmetry of  $L$  iff

$$\int_M L(\tilde{j}'(\tilde{\psi} \circ s)(x)) \tilde{\psi}^*(dx) = \int_M L((js)(x)) dx$$

③ The automorphism  $(\tilde{\psi}, \psi)$  is a symmetry of  $E_u L = 0$  iff  $j^2 \tilde{\psi}$  takes  $so^{1/2} s$  to  $so^{1/2} s$ .

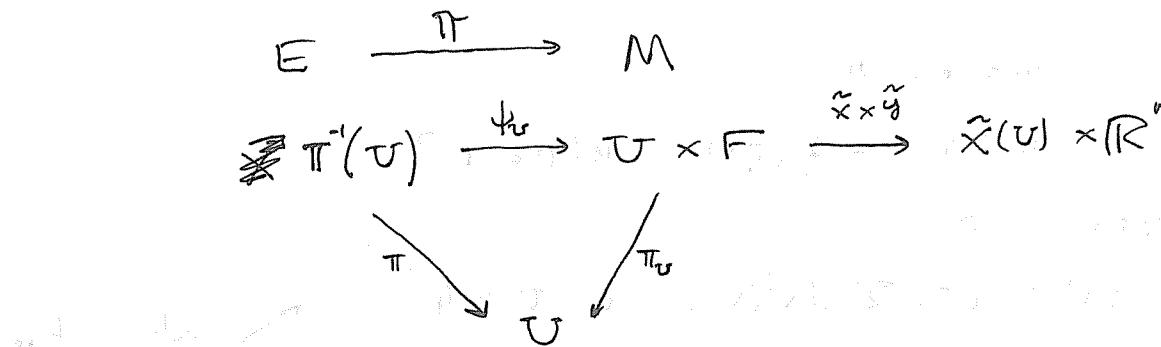
$$\textcircled{1} \Rightarrow \textcircled{2} \Rightarrow \textcircled{3}$$

- We showed that  $L$  was invariant for the Lorentz action so ① holds then ②  $\Rightarrow$  ③ therefore Maxwell's  $E_j^a$ 's ( $so^{1/2} s$  to  $L$ ) are invariant under coordinate change in Lorentzian sense.



# FIBRE BUNDLES

4/30/01  
MA 793



$$\tilde{y}: F \rightarrow \mathbb{R}^n$$

If  $U$  is a chart domain and  $\tilde{x}: U \rightarrow \mathbb{R}^m$  is a chart then you get a chart  $\pi^{-1}(U)$  defined by

$$u \in \pi^{-1}(U) \longrightarrow (\tilde{x}(\pi(u)), \tilde{y}(\pi_F(\psi_U(u))))$$

Usually people are not so formal,

$$x^\mu(u) = \tilde{x}^\mu(\pi(u))$$

$$x^\mu: \pi^{-1}(U) \rightarrow \mathbb{R}$$

$$y^A(u) = \tilde{y}^A(\pi_F(\psi_U(u)))$$

Then we typically just use  $x^\mu$  and  $y^A$ ;  $(x^\mu, y^A)$  components of chart.

②

## Tangent Bundle

4/30/01

$$TM \xrightarrow{\pi} M$$

$(U, \tilde{x})$  chart on  $M$

$$TU = \pi^{-1}(U) = \{(p, v) \in TM \mid p \in U\}$$

$$\pi(p, v) = p$$

$$\psi_v(p, v) = (p, \sum_i d\tilde{x}^i(v) e_i) \in U \times \mathbb{R}^n$$

"adapted"  
chart

$$(T\tilde{x})(p, v) = (\tilde{x}(p), \sum_i d\tilde{x}^i(v) e_i) \in X(U) \times \mathbb{R}^n$$

almost the  
same

$$T_p M$$

## Cotangent Bundle

$$T^*M \xrightarrow{\pi} M$$

$$\psi_v(p, \alpha) = (p, \alpha(\frac{\partial}{\partial \tilde{x}^i}) e^i)$$

$$\psi_v : \mathcal{X}(U) = T^*U \longrightarrow U \times (\mathbb{R}^n)^*$$

$$T^*\tilde{x}$$

$V$  vector space of finite dimension

$$E = M \times V \xrightarrow{\pi} M \quad \pi(p, v) = p$$

$$\psi_m = \text{id}_V$$

These are all "vector bundles" we can add two vectors similarly we can add 2  $\alpha$ 's the fibre is actually a vector space and the map can actually be chosen to be linear map.

Now the ? is what if we want to talk about the Lagrangian on a twisted bundle. He showed us how to make up Jet Bundle for Minkowski space and  $V$ ,  $\dim(V) < \infty$ . The Jet Bundle is diffeomorphic to perhaps  $M \times V \times \mathbb{R}^{n_0}$  we invented symbols for this new bundle to make say  $\frac{\partial}{\partial \dot{x}}$  more concrete.

$$T_p^* M$$

fibres

$$V$$

Def<sup>n</sup> We say  $s: U \rightarrow E$  is a local section of the bundle  $\pi: E \rightarrow M$  iff  $U \subseteq M$  is open as is a smooth map such that  $\pi \circ s = \text{id}_U$

$$\pi(s(p)) = p$$

$$s(p) \in \pi^{-1}(p) = \{u \in E \mid \pi(u) = p\}$$

$$s(p) = s(q) \Rightarrow \pi(s(p)) = \pi(s(q)) \Rightarrow p = q$$

$$d\pi \circ ds = d\text{Id}_U = I$$

thus do. is 1-1

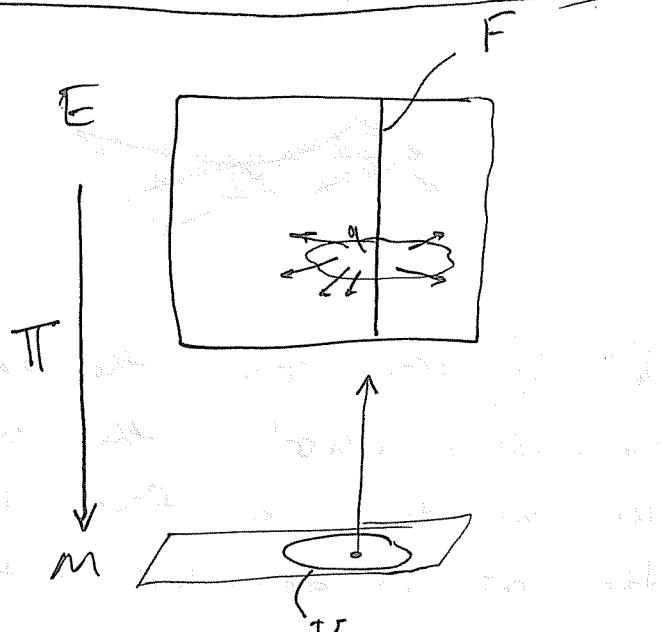
So this is a submanifold mapping

$$v \in T_p M \Rightarrow d_p s(v) \in T_{s(p)} s(U)$$

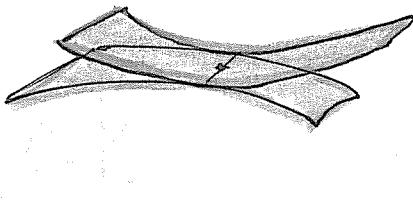
$$ds(T_p M) \subseteq T_{s(p)} s(U) \subseteq T_{s(p)} E$$

Def<sup>n</sup> A vector  $v$  at  $u$  in  $E$  is vertical iff  $d_u \pi(v) = 0$   
a vector is vertical if it lies in the fibre

A local section gives way of decomposing vector space at that point.  $s(U)$  is  $\dim(M)$  while the vertical vectors is  $N$  dimensional. Note that the decomposition is dependent on the choice of section.



We say that two local sections  $s_1$  and  $s_2$  both defined at  $p \in M$  are ~~both~~ jet equivalent iff  $s_1(p) = s_2(p)$  and also  $d_p s_1 = d_p s_2$  (jet equivalent at  $p$ )



Let  $(j^1 s)(p)$  denote the equivalence class (1 jet equivalence class) of the local section  $s$  at  $p$ .  
We can take a fixed local section and see what the set of equivalent sections.

$$J^1 E = \{ (j^1 s)(p) \mid s \in \Gamma_{loc} E, p \in \text{dom}(s) \}$$

This he defines coordinates,

$$x^\mu (j^1 s)(p) = \tilde{x}^\mu(p)$$

$$y^A ((j^1 s)(p)) = \tilde{y}^A(s(p))$$

$$y_\nu^A ((j^1 s)(p)) = \frac{\partial}{\partial \tilde{x}^\mu} (\tilde{y}^A(s(p)))$$

With a little work we could show  $(x^\mu, y^A, y_\nu^A)$  is indeed a chart. We abuse notation a bit (look below).

$$J^1 E \approx (x^\mu, y^A, y_\nu^A) : m + N + mN \text{ dimension}$$



$$E \quad (x^\mu, y^A) : m + N \text{ dimension}$$



$$M \quad \tilde{x}^\mu : m \text{ dimension}$$

technically we ought to have different notation on all levels.

$$(x^r, y^A, y_\mu^A) ((j^1 s)(p)) = \left( \tilde{x}^r(p), \tilde{y}^A(s(p)), \frac{\partial}{\partial x^r} (\tilde{y}^A(s(p))) \right)$$

This is a chart on the 1<sup>st</sup> jet bundle. Fulp works on  $J^\infty E$  a infinite taylor series representation work on  $\infty$  jet bundle but work on it as inverse image of finite jet bundle which is easier then working on  $J^\infty E$  which turns out to be a "Freschet Manifold" [Hamilton in 1981 great paper] develops this stuff from base up.

Defn/ A Lagrangian is a smooth map from  $J^1 E$  into  $\mathbb{R}$

but typically we go into densities instead but that's just a scalar times the volume element unfortunately no time for it now.

$$\frac{\partial L}{\partial u^A} - \partial_\mu^T \left( \frac{\partial L}{\partial u_\mu^A} \right) = 0$$

Given  $L : J^1 E \rightarrow \mathbb{R}$  consider its local coordinate representative

$$J^1 E \xrightarrow{L} \mathbb{R}$$

$$(x^r, y^A, y_\mu^A) \mapsto L(x^r, y^A, y_\mu^A)$$

$$\frac{\partial \bar{L}}{\partial y^A} - \partial_\mu^T \left( \frac{\partial \bar{L}}{\partial y_\mu^A} \right) = \sum_A \bar{L}_A$$

$\bar{L}$  to denote its in local coord.

Now recall what  $\partial_\mu^T$  is,

$$\partial_\mu^T = \frac{\partial}{\partial x^r} + y_\mu^A \frac{\partial}{\partial y^A} + y_{\mu\nu}^A \frac{\partial}{\partial y_\nu^A}$$

$$\sum_A$$

$$\sum_{A,\nu}$$

$$\frac{\partial}{\partial x^\nu} \left( \frac{\partial \bar{L}}{\partial y^A} \left( x^\mu(p), y^A(s(p)), \frac{\partial}{\partial x^\nu} (y^A(s(p))) \right) \right) \rightsquigarrow \partial_\tau$$

Even though  $\bar{L}$  defined on  $T^1 E$  the Euler-Lagrange eq's are in fact defined on  $T^2 E \leftarrow$  but what is  $T^2 E$

Def<sup>1</sup> If  $s_1$  and  $s_2$  are local sections of  $E \rightarrow M$  at  $p \in M$  then  $s_1$  and  $s_2$  are 2-jet equivalent at  $p$  iff

$$s_1(p) = s_2(p)$$

$$d_p s_1 = d_p s_2$$

$$d_p(j^2 s_1) = d_p(j^2 s_2)$$

For our own not making his mistakes  $T^2 E \neq J^1(J^1 E))$

Def<sup>2</sup> Coordinates on  $T^2 E$ ;  $(x^\mu, y^A, y_\mu^A, y_{\mu\nu}^A)$

~~$x^\mu((j^2 s)(p)) = \tilde{x}^\mu(p)$~~

$y^A((j^2 s)(p)) = y^A(s(p))$

$y_\mu^A((j^2 s)(p)) = \frac{\partial}{\partial x^\mu} (y^A(s(p)))$

$y_{\mu\nu}^A((j^2 s)(p)) = \frac{\partial}{\partial x^\nu} \left( \frac{\partial}{\partial x^\mu} (y^A(s(p))) \right)$

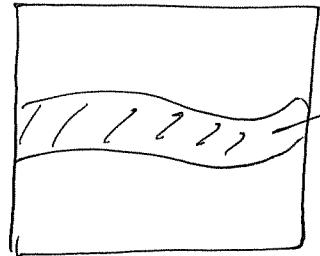
Fact:  $\sum_A \bar{L} : T^2 E \rightarrow \mathbb{R}$

$\uparrow$   
index on  
"contact forms"

"on shell"  $\sum^{(2)} = \{ (j^2 S)(P) \mid \mathcal{E}_A \mathcal{L}((j^2 S)(P)) = 0 \}$

$$\sum^{(2)} = \{ (x^\mu, y^A, y_\nu^{\bar{A}}, y_{\mu\nu}^{\bar{A}}) \mid \mathcal{E}_A \mathcal{L}(x^\mu, y^A, y_\nu^{\bar{A}}, y_{\mu\nu}^{\bar{A}}) = 0 \}$$

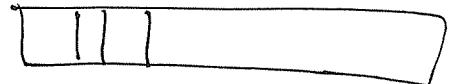
$J^2 E$



$\Sigma^2$

might or might  
not be submanifold

$\sum^{(2)}$  is submanifold if we work on Minkowski



100  
100