

MA 793 I SYMMETRIES IN DIFFERENTIAL EQUATIONS

PROBLEMS: 1.1, 1.6 (Topologists)  
 1.3, 1.4 Alex  
 1.5 Michael  
 1.16 James  
 1.19 Samir  
 1.22 Jason

1/07/01

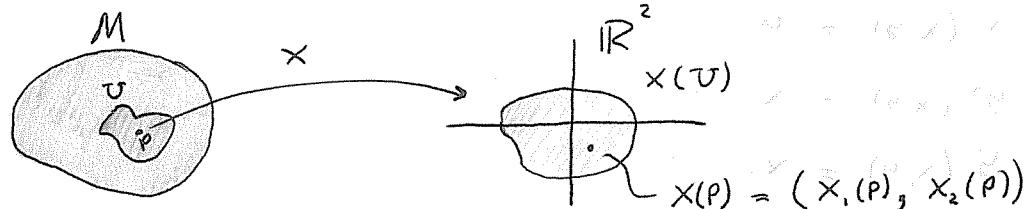
$$\text{For } x = \text{constant}, y = \text{constant}$$

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Def<sup>n</sup> If  $M$  is a set then  $(U, x)$  is a chart on  $M$  iff  $U \subseteq M$  and  $x$  is a bijection from  $U$  onto an open set  $x(U)$  of  $\mathbb{R}^m$  for some  $m$ .



Def<sup>n</sup> If  $(U, x)$  and  $(V, y)$  are charts on  $M$  then  $(U, x)$  and  $(V, y)$  are compatible iff either

$$1.) U \cap V = \emptyset \quad \text{or}$$

$$2.) x(U \cap V), y(U \cap V) \text{ are open and}$$

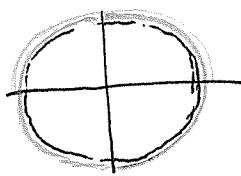
$y \circ x^{-1}: x(U \cap V) \rightarrow y(U \cap V)$  is smooth and has a smooth inverse.

Def<sup>n</sup> If  $M$  is a set then  $\alpha$  is called an atlas for  $M$  iff  $\alpha$  is a family of charts of  $M$  such that

$$1.) \forall p \in M \exists \text{ a chart } (U, x) \text{ in } \alpha \text{ such that } p \in U$$

$$2.) \text{ If } (U, x) \text{ and } (V, y) \text{ are in } \alpha \text{ then they are compatible}$$

Atlas Example  $M = \{(x, y) \mid x^2 + y^2 = 1\}$



$$U^+ = \{(x, y) \in M \mid x > 0\}$$

$$U^- = \{(x, y) \in M \mid x < 0\}$$

$$V^+ = \{(x, y) \in M \mid y > 0\}$$

$$V^- = \{(x, y) \in M \mid y < 0\}$$

Consider Charts  $\{(U^+, x^+), (U^-, x^-), (V^+, y^+), (V^-, y^-)\} = \alpha$

$$x^+(x, y) = y$$

$$x^-(x, y) = y$$

$$y^+(x, y) = x$$

$$y^-(x, y) = x$$

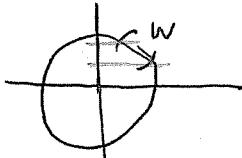
And Now for the inverses

$$(y^+)^{-1}(x) = (x, \sqrt{1-x^2}) \quad \text{with } (y^+)^{-1}(V^+) = (-1, +1)$$

$$[x^+ \circ (y^+)^{-1}](x) = x^+(x, \sqrt{1-x^2}) = \sqrt{1-x^2}$$

Def/<sup>n</sup> If  $\alpha$  is an atlas for a set  $M$ , define  $\alpha^*$  to be the set of all charts  $(W, \beta)$  such that  $(W, \beta)$  is compatible with every chart in  $\alpha$ .

Example



$$\beta = x^+|_W$$

$$x^+ \circ \beta^{-1} = \text{id}_{\beta(W)}$$

Motivates

$$(U, x) \in \alpha$$

$$J \subseteq U.$$

$$x^*(J) = W \quad \text{then } \beta = x|_W \quad (W, \beta)$$

The restriction of a chart is still a chart.

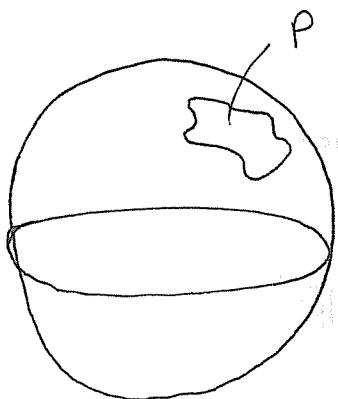
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Proposition

If  $\alpha$  is an atlas so is  $\alpha^*$ .  $\alpha^*$  is called the maximal atlas containing  $\alpha$ :  $\alpha \subseteq \alpha^*$

A maximal atlas is called a differentiable structure for  $M$ .  $(M, \alpha^*)$  is called a manifold

Def<sup>b</sup> If  $(M, \alpha^*)$  is a manifold then  $\Theta \subseteq M$  is open iff for all  $p \in \Theta \exists$  a chart  $(U, x) \in \alpha^*$  and an open set  $W \subseteq x(U)$  such that  $p \in x^{-1}(W) \subseteq \Theta$



$$\downarrow U = \{(x, y, z) | z > 0\}$$

$$\tilde{x}(x, y, z) = (x, y)$$

Proposition : If  $\Theta_1, \Theta_2$  are open so is  $\Theta_1 \cap \Theta_2$ . If  $\{\Theta_\alpha\}$  is a family of open sets so is  $\bigcup \Theta_\alpha$ .

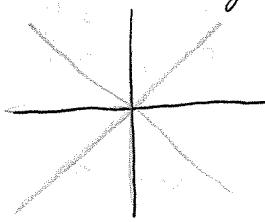
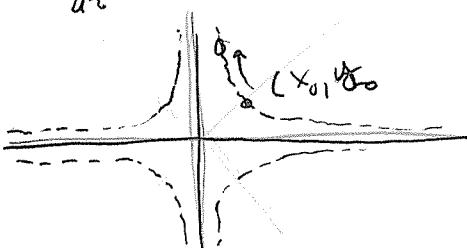
Def<sup>b</sup> A manifold  $M$  is Hausdorff if  $\forall p, q \in M, p \neq q \exists$  open sets  $\Theta_p, \Theta_q$  such that  $p \in \Theta_p, q \in \Theta_q$  and  $\Theta_p \cap \Theta_q = \emptyset$

Example Non-Hausdorff manifolds

$$\left. \begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x \end{aligned} \right\} \rightarrow \frac{dy}{dx} = \frac{x}{y} \rightarrow y dy = x dx$$

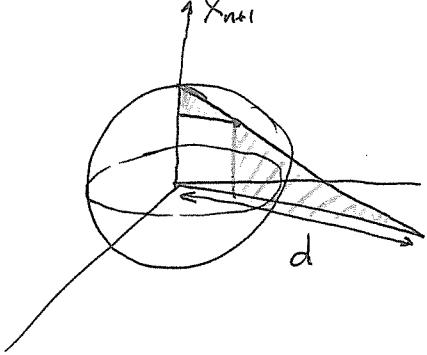
$$\frac{1}{2} y^2 = \frac{1}{2} x^2 + C$$

$$y^2 - x^2 = K$$



Manifold Structure on the  $n$  dimensional sphere

$$(x_1, x_2, \dots, x_n, x_{n+1}) = (\vec{x}, x_{n+1})$$



$$\frac{1}{d} = \frac{1 - x_{n+1}}{\|\vec{x}\|}$$

$$z(x_1, x_2, \dots, x_{n+1}) = d \frac{\vec{x}}{\|\vec{x}\|} = \frac{d}{\|\vec{x}\|} \vec{x}$$

$$z(x_1, x_2, \dots, x_{n+1}) = \left( \frac{1}{1 - x_{n+1}} \right) (x_1, x_2, \dots, x_n)$$

$$z : \{x \in S^n \mid x_{n+1} \neq 1\} \rightarrow \mathbb{R}^n$$

GRADE BASED ON JUST HOMEWORK AND PROJECT

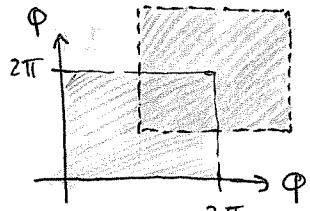
THE TAURUS

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$$T^2 = \{(e^{i\theta}, e^{i\varphi}) \mid \theta, \varphi \in \mathbb{R}\} \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$$

$$(\theta, \varphi) \xrightarrow{\chi^{-1}} (e^{i\theta}, e^{i\varphi})$$

$$\mathbb{R}^2 \rightarrow T^2$$



$$0 < \theta < 2\pi$$

$$0 < \varphi < 2\pi$$



$$\chi_1(e^{i\theta}, e^{i\varphi}) = (\theta, \varphi)$$

$$\chi_2^{-1} : (\theta, \varphi) \rightarrow (e^{i\theta}, e^{i\varphi})$$

$$\pi < \theta < 3\pi$$

$$\pi < \varphi < 3\pi$$

we map the boxes onto the Taurus  $T^2$

$$(\chi_2 \circ \chi_1^{-1})(\theta, \varphi) = (\theta + \pi, \varphi + \pi)$$

Notice that  $T^2 = S^1 \times S^1$  thus we motivate the general torus  $T^n = \underbrace{S^1 \times S^1 \times \dots \times S^1}_n$  with

$$x^*(\theta_1, \theta_2, \dots, \theta_n) = (e^{i\theta_1}, \dots, e^{i\theta_n})$$

Could show  $T^n$  a manifold by arguing cartesian product of manifold is a manifold or by finding charts.

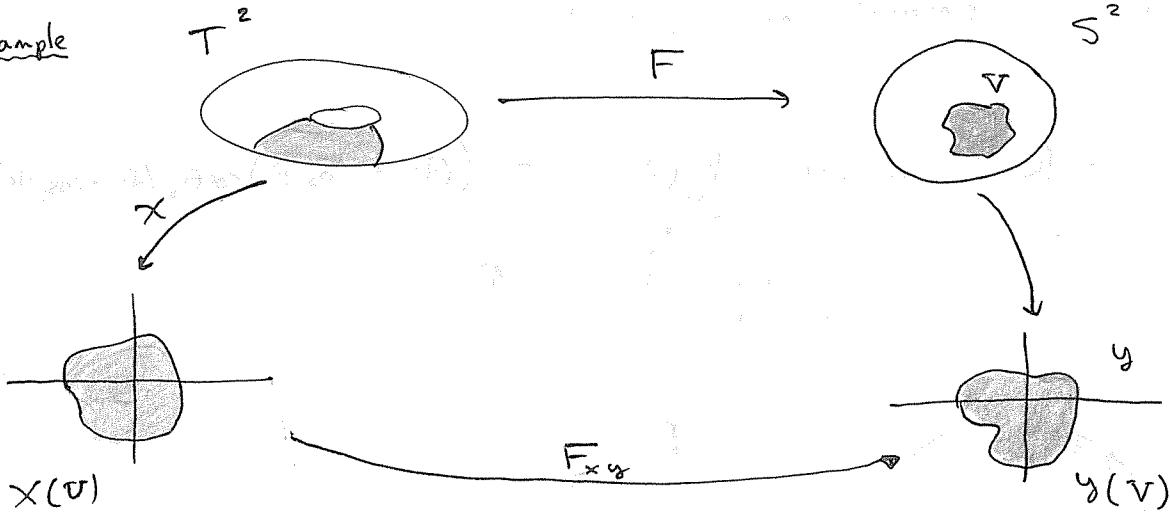
Defn If  $M$  and  $N$  are manifolds and  $F: M \rightarrow N$  then we say that  $F$  is smooth iff

forall pairs of charts  $(U, x)$  on  $M$  and  $(V, y)$  on  $N$  the function  $F_{xy} = y \circ F \circ x^{-1}$

$$F_{xy} = y \circ F \circ x^{-1}$$

THIS  $F_{xy}$  is the local coordinate representation of  $F$

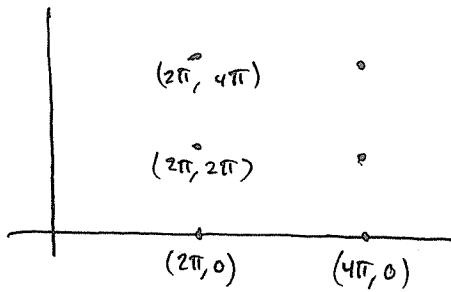
Example



$$F_{xy}: x(U \cap F^{-1}(V)) \rightarrow y(V)$$

Example : LATTICE IN THE PLAIN

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$$(\theta, \varphi) \mapsto (\lambda^{i\theta}, \lambda^{i\varphi})$$

$$\mathbb{R}^2 \longrightarrow T^2$$

$$\frac{\mathbb{R}}{2\pi\mathbb{Z}} \cong S^1$$

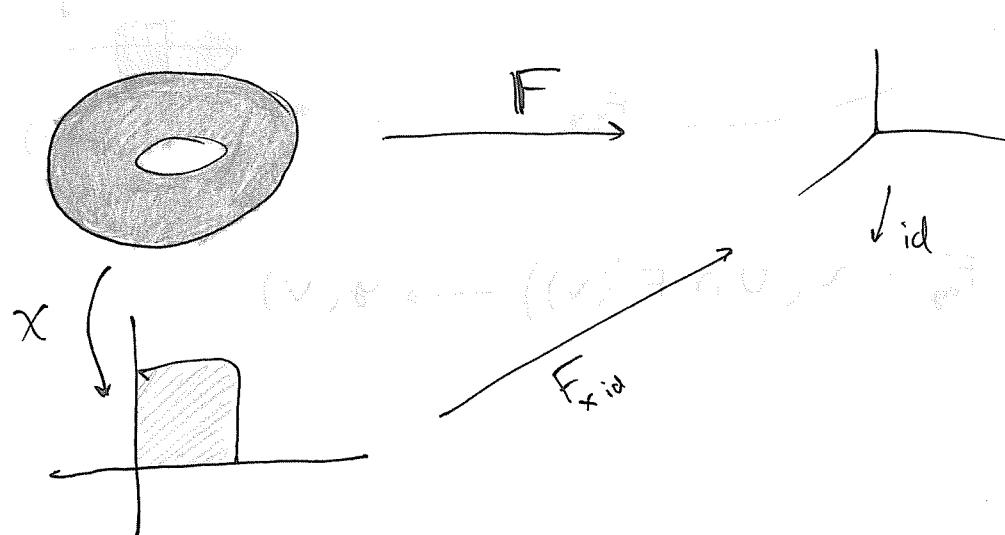
$$\left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right) \times \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)$$

THEOREM

$f: T^2 \rightarrow N$  is smooth iff  $\exists \tilde{f}: \mathbb{R}^2 \rightarrow N$  which is smooth and periodic

$$T^2 \longrightarrow \mathbb{R}^3 \text{ where } F(\theta, \varphi) = ((\sqrt{2} + \cos \varphi) \cos \theta, (\sqrt{2} + \cos \varphi) \sin \theta, \sin \varphi)$$

↑  
one of maps identity on  $\mathbb{R}^3$   
from before



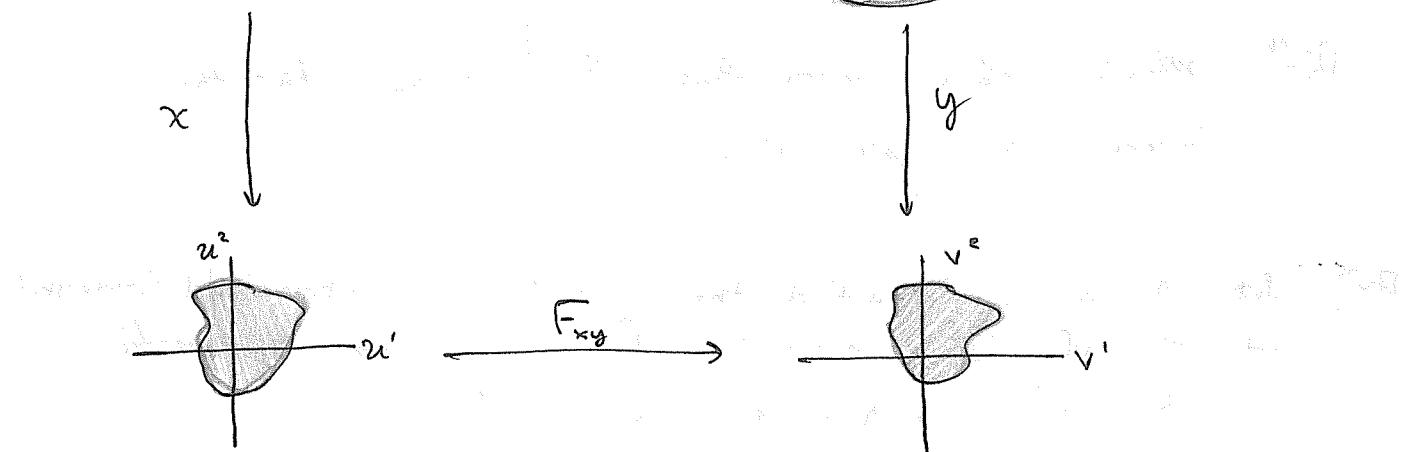
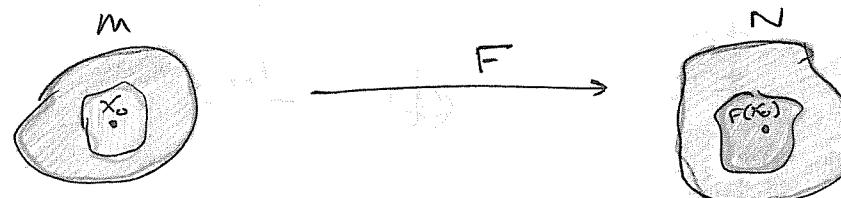
Assume  $F: M \rightarrow N$  is smooth. Then for  $x_0 \in M$  the rank of

$F$  at  $x_0 \in M$  is defined to be the rank of  $J_{F_{xy}}(x(x_0))$  for

some pair of charts  $(U, x)$  such that  $x_0 \in U$ ,

$(V, y)$  such that  $F(x_0) \in V$ . Note that in the book

$$\frac{\partial F^j}{\partial x^i} = \frac{\partial}{\partial u^i} (y^j \circ F \circ x^{-1})$$



$J_{F_{xy}}(x_0)$  is an  $m \times n$  matrix for which we can find the rank in the usual ways.

Now is the rank chart dependent?

if  $(U, x)$  and  $(\bar{U}, \bar{x})$  are charts at  $x_0$

if  $(V, y)$  and  $(\bar{V}, \bar{y})$  are charts at  $y_0 = F(x_0)$

$$\begin{aligned} F_{\bar{x}\bar{y}} &= \bar{y} \circ \bar{F} \circ \bar{x}^{-1} = (\bar{y} \circ y^{-1}) \circ (y \circ F \circ x^{-1}) \circ (x \circ \bar{x}^{-1}) \\ &= (\bar{y} \circ y^{-1}) \circ F_{xy} \circ (x \circ \bar{x}^{-1}) \end{aligned}$$

$$J_{\bar{x}\bar{y}} = J_{\bar{y} \circ y^{-1}} J_{F_{xy}} J_{x \circ \bar{x}^{-1}}$$

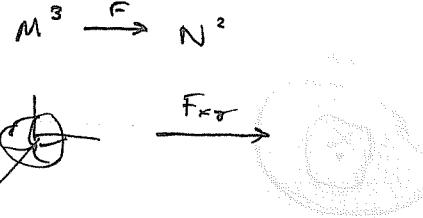
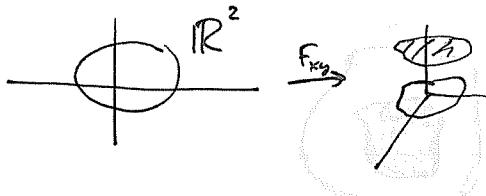
Then by theorem from L.A.  
rank is preserved by such a transform

THEOREM

Let  $F: M \rightarrow N$  be smooth and assume  $F$  has maximal rank at  $x_0 \in M$ . Then  $\exists$  charts  $(U, \varphi)$  at  $x_0$  and  $(V, \psi)$  at  $F(x_0)$  such that

- 1.)  $n > m \Rightarrow F_{xy}(u^1, \dots, u^m) = (u^1, u^2, \dots, u^m, 0, \dots, 0)$
- 2.)  $n \leq m \Rightarrow F_{xy}(u^1, u^2, \dots, u^m) = (u^1, u^2, \dots, u^m)$

$$F: M^2 \rightarrow N^3$$



Def<sup>b</sup>/ Maximal Rank means that  $\frac{\partial F^j}{\partial x^i}(x(x_0))$  has the largest rank possible.

Def<sup>b</sup>/ Let  $M$  be a manifold then  $N \subseteq M$  is a submanifold (immersed) of  $M$  iff  $\exists$  a manifold  $\tilde{N}$  and a smooth function

$\varphi: \tilde{N} \rightarrow N \subseteq M$  such that

1.)  $\varphi$  is injective.

2.)  $\varphi$  satisfies the maximal rank condition at every point.

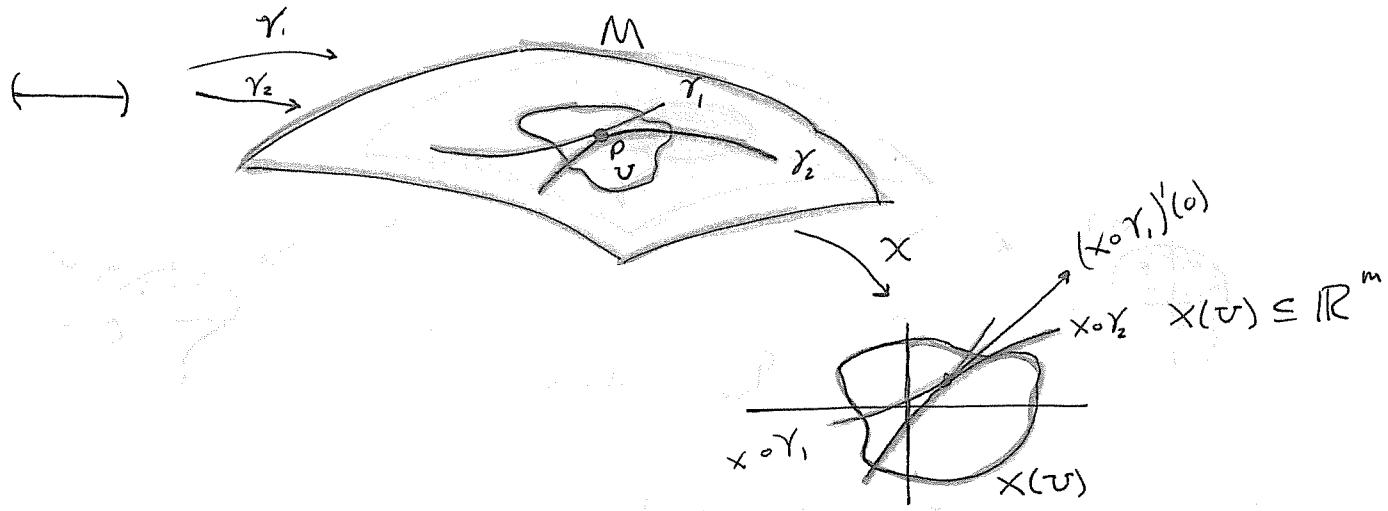
How do we define a tangent space. What is the manifold imbedded in, does it matter? See below 2 embeddings



of the circle, one in  $\mathbb{R}^2$  one in  $\mathbb{R}^3$ . We would like the imbedding not to matter...

**Defn** Let  $M$  be a Manifold and  $p \in M$ . If  $\gamma_1 : (-a_1, a_1) \rightarrow M$  and  $\gamma_2 : (-a_2, a_2) \rightarrow M$  are smooth maps such that  $\gamma_1(0) = p = \gamma_2(0)$  then we say  $\gamma_1 \sim \gamma_2$  iff  $\exists$  a chart  $(U, x)$   $\ni p \in U$  and  $(x \circ \gamma_1)'(0) = (x \circ \gamma_2)'(0)$

— THIS DEFINES THE  $p$  EQUIVALENCE —



$$\frac{d}{dt} \left\{ x^1(\gamma_1(t)), x^2(\gamma_1(t)), \dots, x^m(\gamma_1(t)) \right\} = (x \circ \gamma_1)'(t)$$

$$\frac{d}{dt} \left[ x^i(\gamma_1(t)) \right]_{t=0} = \frac{d}{dt} \left[ x^i(\gamma_2(t)) \right]_{t=0}$$

Def<sup>b</sup>/ A vector tangent to  $M$  at  $p$  is an equivalence class of curves  $[\gamma]_p$ . If  $v = [\gamma]_p \in T_p M$

Then the components of  $v$  relative to  $(U, x)$  are

$$v_x^i = \left. \frac{d}{dt} (x^i(\gamma(t))) \right|_{t=0}$$

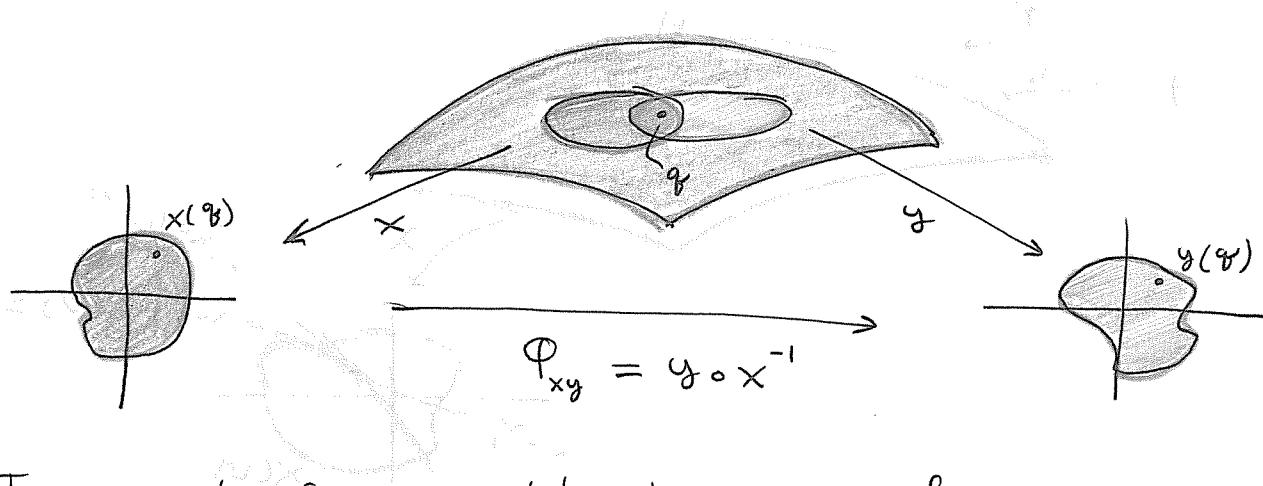
We physicists often write vectors via components  $v^i$  or  $\bar{v}^i$   
how do we know  $v^i$  and  $\bar{v}^i$  are the same, that depends  
on the group of acceptable transformations. Usually

$$\bar{v}^i = \Theta^i_j v^j$$

Or in more mathy notation

$$\bar{v} = \Theta v$$

How do the components of  $v = [\gamma]_p$  change when you  
change coordinates? Let  $(U, x)$  and  $(V, y)$  be charts  
with  $p \in U \cap V$ .



The components of  $v$  relative to  $x$  are of course

$$v_x^i = \left. \frac{d}{dt} (x^i(\gamma(t))) \right|_{t=0}$$

The components of  $v$  relative to  $y$  are likewise

$$v_y^i = \left. \frac{d}{dt} (y^i(\gamma(t))) \right|_{t=0}$$

(We could write  $v_x = (x \circ \gamma)'(0)$  and  $v_y = (y \circ \gamma)'(0)$  to avoid components)

Let  $\Phi_{xy} = y \circ x^{-1}$ . Claim:

$$(\mathcal{D}\Phi_{xy})(v_x) = v_y$$

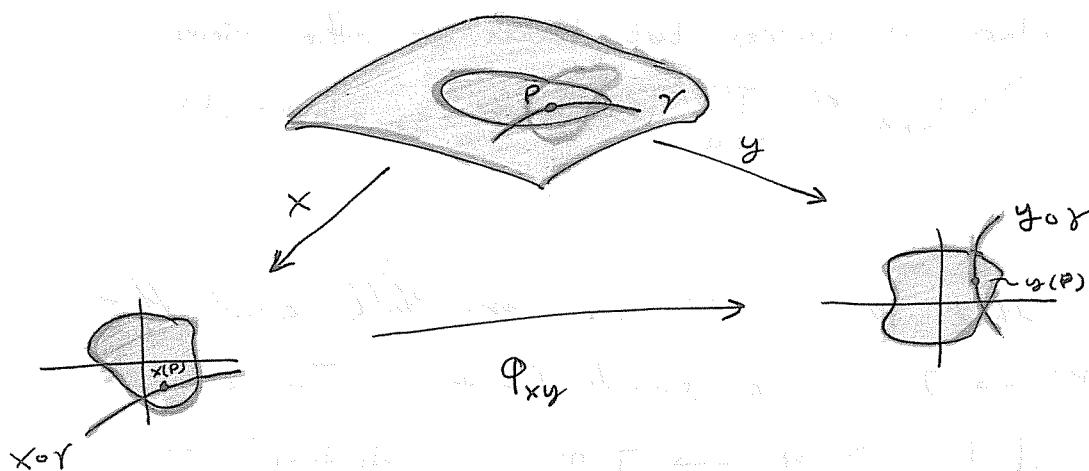
$$\text{Notice } \forall q \in U \cap V \quad \Phi_{xy}(x(q)) = (y \circ x^{-1})(x(q)) = y(q)$$

$$\therefore \Phi_{xy}(x(\gamma(t))) = y(\gamma(t))$$

So we may write out the following

$$\Phi_{xy}(x^1(\gamma(t)), x^2(\gamma(t)), \dots, x^m(\gamma(t))) = (y^1(\gamma(t)), \dots, y^m(\gamma(t)))$$

Again the picture for reference



We can differentiate on  $\mathbb{R}^m$  with less difficulty

$$y^i(\gamma(t)) = \Phi_{xy}^i(x^1(\gamma(t)), x^2(\gamma(t)), \dots, x^m(\gamma(t)))$$

$$\frac{d}{dt}(y^i(\gamma(t))) = \sum_{i=1}^m \frac{\partial \Phi_{xy}^i}{\partial u^i} \frac{d}{dt}(x^i(\gamma(t)))$$

$$v_y^i = \sum_{i=1}^m \frac{\partial \Phi_{xy}^i}{\partial u^i} v_x^i$$

$$\text{Note: } \Phi_{xy}(u) = y(x^{-1}(u)) \Rightarrow \frac{\partial(y^i \circ x^{-1})}{\partial u^i}$$

$$\partial_i \Phi_{xy}^i = \frac{\partial \Phi_{xy}^i}{\partial u^i} = \frac{\partial(y^i \circ x^{-1})}{\partial u^i} = \underline{\underline{\frac{\partial y^i}{\partial x^i}}}$$

Shorthand for LHS

$$V_y^T = J_{\Phi_{xy}}(x(p)) V_x^T$$

$$V_y^i = \frac{\partial y^i}{\partial x^j} V_x^j$$

$$V_y^i = \sum_{j=1}^m \frac{\partial \Phi_{xy}^j}{\partial x^i} V_x^j$$

$$V_y = V_x J_{\Phi_{xy}}(x(p))^T$$

So what is a vector. In here we shall think of it as an equivalence class of curves but there are other views.

$$(V_x)_{x \in A} \in \prod_{x \in A} \mathbb{R}^m \quad (\text{crazy algebra})$$

———— //

Assume that  $M$  and  $N$  are manifolds and that

$F: M \rightarrow N$  is a smooth function. If  $p \in M$

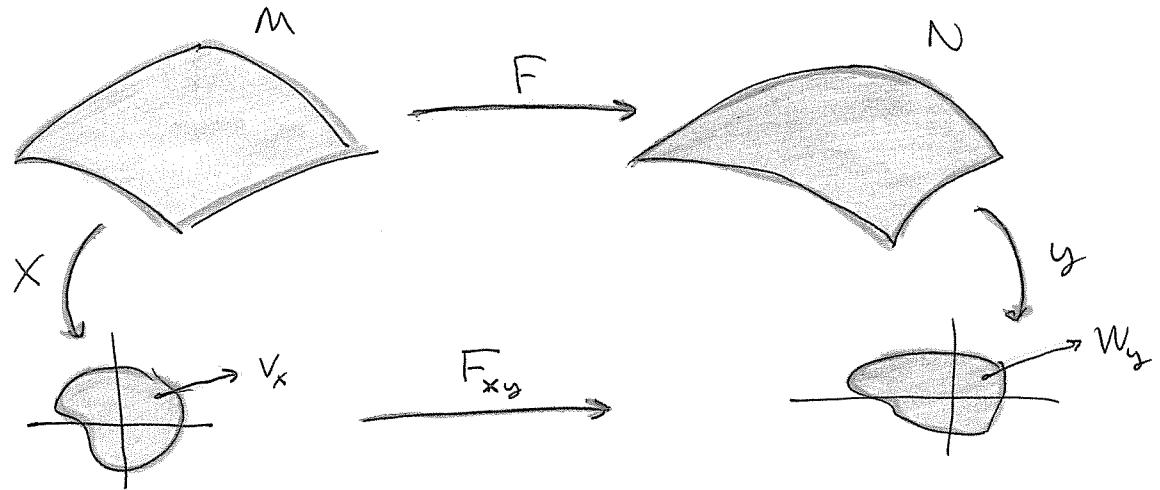
then  $d_p F: T_p M \rightarrow T_{F(p)} N$  is defined as

follows : If  $v = [\gamma]_p$  then  $d_p F(v) = [F \circ \gamma]_{F(p)}$

QUESTION : If  $V_x^i$  are the components of  $v = [\gamma]$  in a chart  $x$ . Show how they relate to the components of  $d_p F(v)$  in a chart  $y$ ?

Let  $w = [F \circ \gamma]_{F(p)}$

$$w^i = \left. \frac{d}{dt} \{y^i(F(\gamma(t)))\} \right|_{t=0}$$



$F_{xy}$  is the local coordinate representative of  $F$ .

$$\begin{aligned}
 w_y^i &= \frac{d}{dt} \left\{ y^i(F(r(t))) \right\}_{t=0} \\
 &= \frac{d}{dt} \left\{ (y^i \circ F \circ x^{-1})(x(r(t))) \right\}_{t=0} \\
 &= \frac{d}{dt} \left\{ F_{xy}^i(x^1(r(t)), x^2(r(t)), \dots, x^m(r(t))) \right\}_{t=0} \\
 &= \sum_{i=1}^m \frac{\partial F_{xy}^i}{\partial u^i} \frac{d(x^i \circ r)(t)}{dt} \Big|_{t=0} \\
 &= \sum_{i=1}^m J_{F_{xy}}(x(p))_i^j v_x^i
 \end{aligned}$$

So without the coordinates in plain row notation,

$$w_y = v_x J_{F_{xy}}(x(p))^T$$

Exercise : Chain Rule  $d(G \circ F) = dG \circ dF$

$$\begin{aligned}
& \left( \frac{\partial}{\partial x} \right)^m \left( \frac{\partial}{\partial y} \right)^n \left( \frac{\partial}{\partial z} \right)^p \left( \frac{\partial}{\partial w} \right)^q \left( \frac{\partial}{\partial v} \right)^r \left( \frac{\partial}{\partial u} \right)^s \\
& \times \left( \frac{\partial}{\partial x} \right)^{m'} \left( \frac{\partial}{\partial y} \right)^{n'} \left( \frac{\partial}{\partial z} \right)^{p'} \left( \frac{\partial}{\partial w} \right)^{q'} \left( \frac{\partial}{\partial v} \right)^{r'} \left( \frac{\partial}{\partial u} \right)^{s'} \\
& \times \left( \frac{\partial}{\partial x} \right)^{m''} \left( \frac{\partial}{\partial y} \right)^{n''} \left( \frac{\partial}{\partial z} \right)^{p''} \left( \frac{\partial}{\partial w} \right)^{q''} \left( \frac{\partial}{\partial v} \right)^{r''} \left( \frac{\partial}{\partial u} \right)^{s''} \\
& \times \left( \frac{\partial}{\partial x} \right)^{m'''} \left( \frac{\partial}{\partial y} \right)^{n'''} \left( \frac{\partial}{\partial z} \right)^{p'''} \left( \frac{\partial}{\partial w} \right)^{q'''} \left( \frac{\partial}{\partial v} \right)^{r'''} \left( \frac{\partial}{\partial u} \right)^{s'''} \\
& \times \left( \frac{\partial}{\partial x} \right)^{m''''} \left( \frac{\partial}{\partial y} \right)^{n''''} \left( \frac{\partial}{\partial z} \right)^{p''''} \left( \frac{\partial}{\partial w} \right)^{q''''} \left( \frac{\partial}{\partial v} \right)^{r''''} \left( \frac{\partial}{\partial u} \right)^{s''''} \\
& \times \left( \frac{\partial}{\partial x} \right)^{m'''''} \left( \frac{\partial}{\partial y} \right)^{n'''''} \left( \frac{\partial}{\partial z} \right)^{p'''''} \left( \frac{\partial}{\partial w} \right)^{q'''''} \left( \frac{\partial}{\partial v} \right)^{r'''''} \left( \frac{\partial}{\partial u} \right)^{s'''''}
\end{aligned}$$

Review

Curves  $\gamma_1, \gamma_2$  through  $p$  are equivalent  $\gamma_1 \sim \gamma_2$  iff  $(x \circ \gamma_1)'(0) = (x \circ \gamma_2)'(0)$   
for some chart  $x$  (pgs 24, 25)

Def<sup>a</sup>/ Tangent vector is an equivalence class  $v = [\gamma] \in T_p M$

Def<sup>b</sup>/ If  $v = [\gamma] \in T_p M$ ,  $(U, x)$  is a chart such that  $p \in U$  then the components of  $v$  relative to  $x$  are

$$v_x = ((x^1 \circ \gamma)'(0), (x^2 \circ \gamma)'(0), \dots, (x^m \circ \gamma)'(0))$$

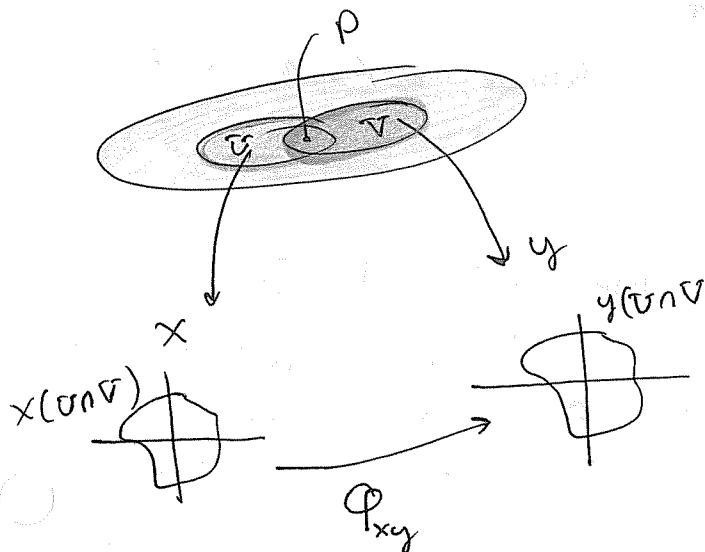
Def<sup>c</sup>/ If  $(U, x)$  and  $(V, y)$  are charts at  $p$  and  $\varphi_{xy} = y \circ x^{-1}$

then  $d_{x(p)} \varphi_{xy}(v_x) = v_y$  or  $v_x J_{\varphi_{xy}}(x(p))^T$

Def<sup>d</sup>/ If  $F: M \rightarrow N$  and  $v = [\gamma] \in T_p M$  then (pages 32, 33)

$d_p F(v) = (F \circ \gamma) = w$ . Furthermore if  
 $(U, x)$  and  $(V, y)$  are charts in  $M, N$  then

$$d_{x(p)} F_{xy}(v_x) = v_x J_{F_{xy}}(x(p))^T = w_y$$



If one is given a tangent vector  $v = [\gamma]_p \in T_p W$  where  $W \subseteq \mathbb{R}^m$  is open then one a vector space isomorphism

$$[\gamma] \rightarrow \left( \frac{d\gamma^1}{dt}(0), \dots, \frac{d\gamma^m}{dt}(0) \right)$$

$$T_p W \rightarrow \mathbb{R}^m$$

$$D\varphi_{xy}(v_x) = v_y \iff d_{x(p)} \varphi_{xy}(v_x) = v_y$$

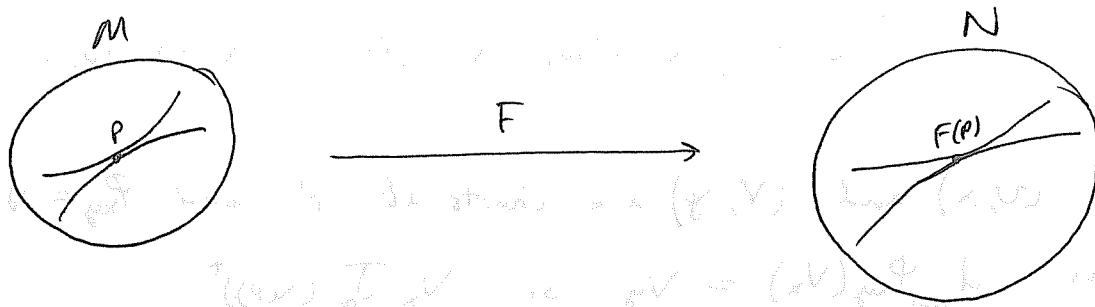
Vector  $v^i$  say under  $x$  coordinates  
 vector  $\bar{v}^i$  would be under  $\bar{x}$  coordinates

Typical physics notation

$$v^i \frac{\partial \bar{x}^j}{\partial x^i} = \bar{v}^j \quad \text{shorthand for } \partial_i(\bar{x}_0 \bar{x}^j)$$

sometimes see  $\bar{x} = \Phi(x)$

just like  $\Phi_{xy} = y_0 x^{-1}$



$d_p F : T_p M \rightarrow T_{F(p)} N$  maps equivalence classes of curves in  $T_p M$  to equivalence classes of curves in  $T_{F(p)} N$

Now if  $F : M \rightarrow \mathbb{R}^n$  then (we) are free to make the identification  $T_q \mathbb{R}^n \cong \mathbb{R}^n$ . Thus

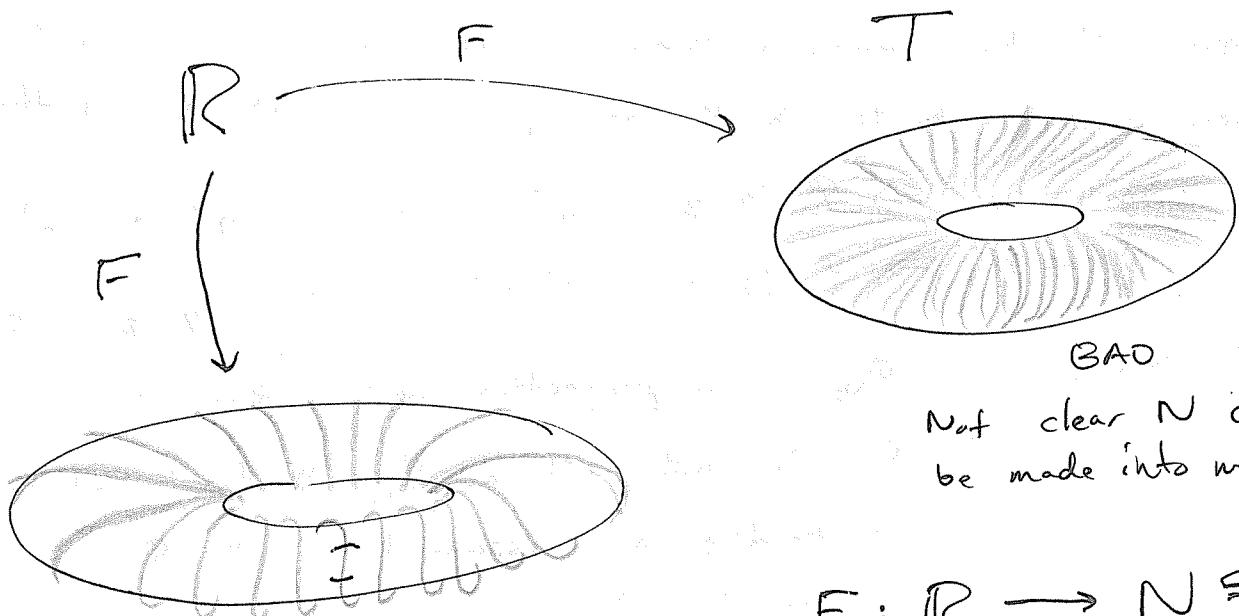
$$d_p F : T_p M \rightarrow T_{F(p)} \mathbb{R}^n \cong \mathbb{R}^n$$

Particularly if  $n=1$ ,  $d_p F \in (T_p M)^*$  because  $dF : M \rightarrow \mathbb{R}$

Def<sup>n</sup>/  $N \subseteq M$  is an (immersed) submanifold iff  $\exists$

a manifold  $\tilde{N}$  and a map  $\Phi : \tilde{N} \rightarrow N \subseteq M$  s.t.

- 1.)  $\Phi : \tilde{N} \rightarrow N$  is bijective,  $\Phi : \tilde{N} \rightarrow M$  is smooth
- 2.) the rank of  $\Phi$  is maximal at each point of  $\tilde{N}$

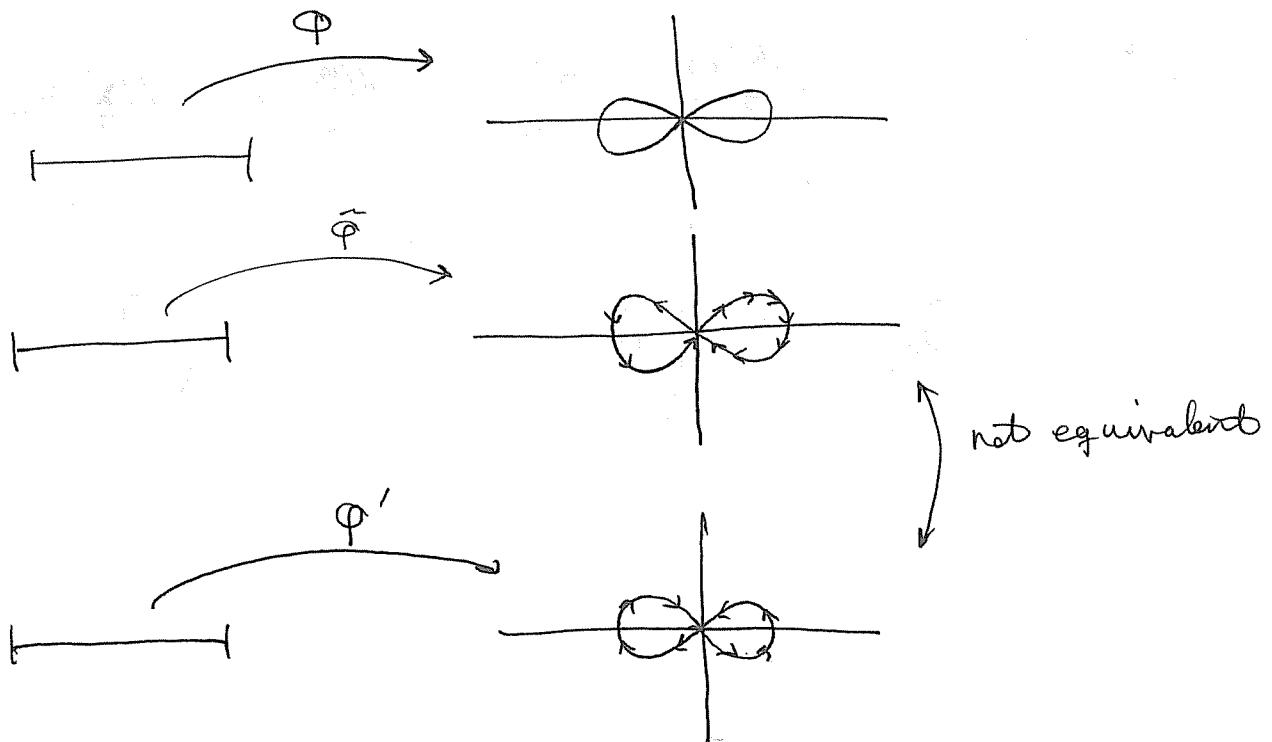


Not clear  $N$  can  
be made into manifold

$$F: R \rightarrow N \subseteq T$$

NICE we can make  
 $N$  into a manifold  
with structure compatible  
to  $M$

Proposition : Assume  $\varphi: \tilde{N} \rightarrow N \subseteq M$  is a bijection  
onto  $N$  and is smooth. Then  $\varphi$  has maximal  
rank at each point of  $\tilde{N}$  iff  $d_p \varphi: T_p \tilde{N} \rightarrow T_{\varphi(p)} M$   
is injective for all  $p \in \tilde{N}$ .



Proof/ Assume  $\varPhi$  has maximal rank. Choose any point  $p \in \tilde{N}$   
 There exist charts  $x, y$  at  $p$  and at  $\varPhi(p)$  such that

$$m \geq n \Rightarrow \varPhi_{xy}(u^1, u^2, \dots, u^n) = (u^1, u^2, \dots, u^n, 0, 0, \dots, 0) \quad (1)$$

$$m < n \Rightarrow \varPhi_{xy}(u^1, u^2, \dots, u^m, u^{m+1}, \dots, u^n) = (u^1, u^2, \dots, u^m)$$

But  $m < n \Rightarrow \varPhi_{xy}$  is a projection and is thus not 1-1

$\Rightarrow \varPhi$  is not 1-1 as  $\varPhi_{xy} = y \circ \varPhi_{xy}^{-1}$  which contradicts our assumption on  $\varPhi$ .

$$\therefore m \geq n$$

Assume that  $d_p \varPhi(V_1) = d_p \varPhi(V_2)$ . We show  $V_1 = V_2$ .

Let  $v_1 = [\gamma_1], v_2 = [\gamma_2]$  then

$$v_{1x}^i = \frac{d(x^i \circ \gamma_1)}{dt}(0) \quad v_{2x}^i = \frac{d(x^i \circ \gamma_2)}{dt}(0)$$

$$\Leftrightarrow d\varPhi_{xy}(v_{1x}) = d\varPhi_{xy}(v_{2x})$$

$$v_{1x} J_{\varPhi_{xy}}^T = v_{2x} J_{\varPhi_{xy}}^T$$

$$\left( \frac{d(x^1 \circ \gamma_1)}{dt}(0), \dots, \frac{d(x^n \circ \gamma_1)}{dt}(0) \right) J_{\varPhi_{xy}}^T = \left( \frac{d(x^1 \circ \gamma_2)}{dt}(0), \frac{d(x^2 \circ \gamma_2)}{dt}(0), \dots, \frac{d(x^n \circ \gamma_2)}{dt}(0) \right) J_{\varPhi_{xy}}^T$$

Now examine the Jacobian of  $\varPhi_{xy}$  for  $m \geq n$

$$J_{\varPhi_{xy}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$$

$$\left( \frac{d(x'_0 v_1)}{dt}, \dots, \frac{d(x''_0 v_1)}{dt} \right) [I_n \cdot 0] = \left( \frac{d(x'_0 v_2)}{dt}, \dots, \frac{d(x''_0 v_2)}{dt} \right) [I_n \cdot 0]$$

$$\begin{aligned} & \quad \parallel & & \quad \parallel \\ (\vec{v}_{1x}, 0, \dots, 0) &= (\vec{v}_{2x}, 0, \dots, 0) \\ \therefore \vec{v}_{1x} &= \vec{v}_{2x} \quad \therefore v_1 = v_2 \end{aligned}$$

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### Proposition

Assume  $\varphi: \tilde{N} \rightarrow N \subset M$  is a smooth map which is injective and  $N = \varphi(\tilde{N})$ . THEN  $\varphi$  has maximal rank at each  $p \in \tilde{N}$  iff  $d\varphi_p: T_p \tilde{N} \rightarrow T_{\varphi(p)} M$  is injective.

Recall from last time we showed

maximrank rank of  $\varphi \Rightarrow d\varphi$  injective

$$d\varphi_p[\gamma_1] = d\varphi_p[\gamma_2] \quad \gamma_1(0) = p = \gamma_2(0)$$

$$\frac{d(x \circ \gamma_1)}{dt}(0) = \frac{d(x \circ \gamma_2)}{dt}(0)$$

$$d_x[\gamma_1] = d_x[\gamma_2]$$

$$\gamma_1 \approx \gamma_2$$

$$[\gamma_1] = [\gamma_2]$$

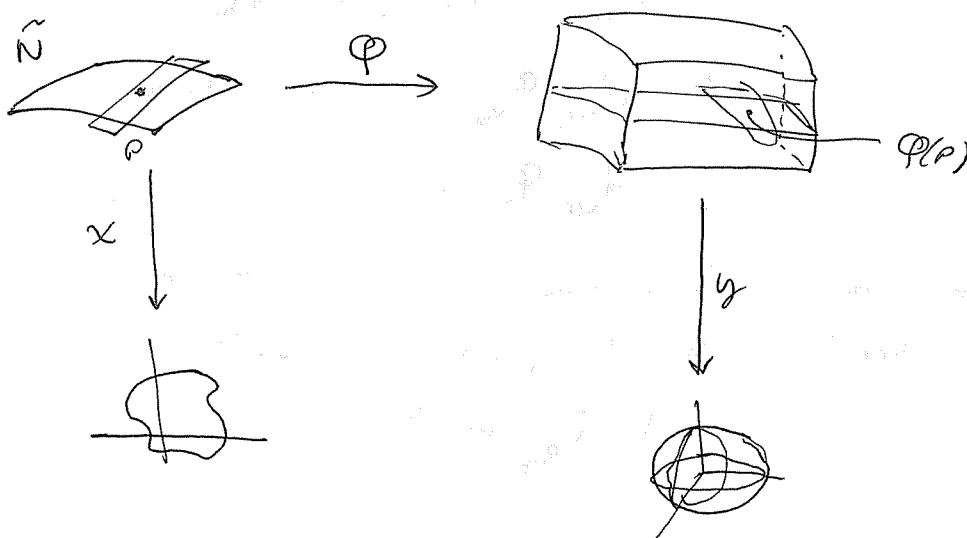
$\rightarrow$  better end to previous argument

~~//~~

$d\varphi$  is injective  $\Rightarrow$  max rank of  $\varphi$

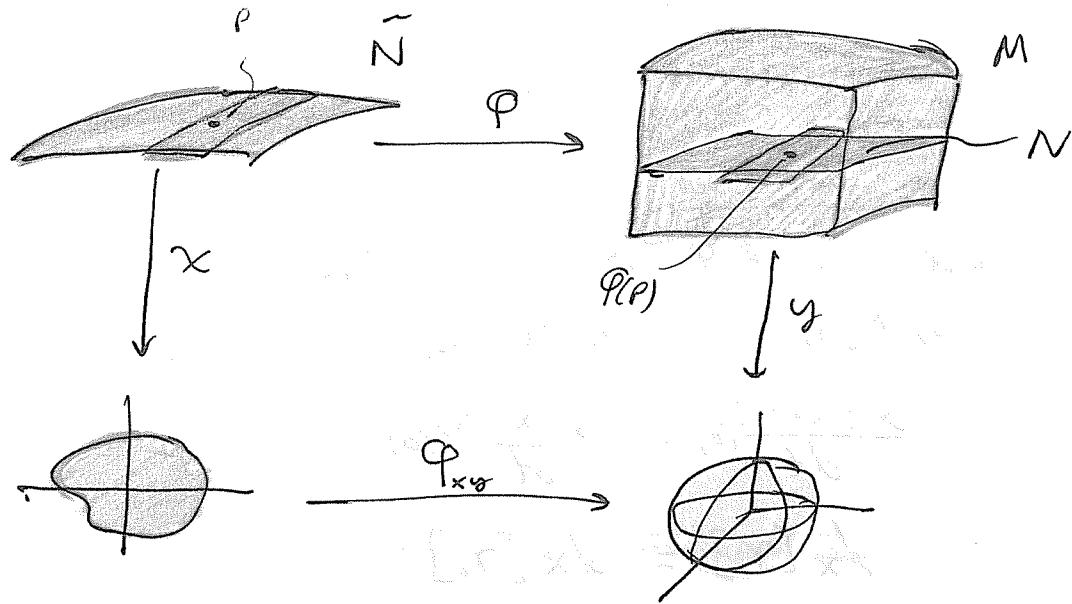
Assume  $p \in \tilde{N}$  and  $d\varphi_p: T_p \tilde{N} \rightarrow T_{\varphi(p)} M$  is injective

We show  $\varphi$  has maximal rank at  $p$ .



(Repeated on next page)

Assume  $p \in \tilde{N}$  and  $d_p \varphi : T_p \tilde{N} \rightarrow T_{\varphi(p)} M$  is injective  
 We show  $\varphi$  has maximal rank at  $p$



Choose a chart  $(U, x)$  in  $\tilde{N}$  such that  $p \in U$   
 and a chart  $(V, y)$  of  $M$  with  $\varphi(p) \in V$   
 We show rank of  $J_{\varphi_{xy}}(x(p))$  is maximal

$$\varphi_{xy} \circ x = y \circ \varphi$$

$$d_{x(p)} \varphi_{xy} \circ d_p x = d_{\varphi(p)} y \circ d_p \varphi$$

Dropping some notation

$$d \varphi_{xy} = d y \circ d \varphi \circ (d x)$$

$\therefore d_{x(p)} \varphi_{xy}$  is injective

$$\therefore d_{x(p)} \varphi_{xy}(v) = 0 \Rightarrow v = 0$$

(Because a linear map is injective  $\Leftrightarrow$  nullspace of map is  $\{0\}$ .)

Likewise with the jacobian we get

$$V_x J_{\varphi_{xy}}(x(p))^T = 0 \Rightarrow V_x = 0$$

$$J_{\varphi_{xy}}(x(p)) V_x^T = 0 \Rightarrow V_x^T = 0$$

Suppose  $A$  is an  $m \times n$  matrix such that  $Aw = 0 \Rightarrow w = 0$   
 Claim :  $A$  has maximal rank,

$$\begin{pmatrix} A^1 & A^2 & \cdots & A^n \\ A_1 & A_2 & \cdots & A_n \\ \vdots & \vdots & \ddots & \vdots \\ A_m & A_m & \cdots & A_m \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Changing Notation slightly  $A^{(1)}$  the (1) row

$$\Leftrightarrow w_1 A^{(1)} + w_2 A^{(2)} + \cdots + w_n A^{(n)} = \vec{0} \Rightarrow w_i = 0 \quad \forall i$$

$\Rightarrow$  columns  $A^{(1)}, A^{(2)}, \dots, A^{(n)}$  are linearly independent

$\therefore \text{rank}(A) = n \therefore A$  has maximal rank.

QED

### Summary

$\tilde{N}, M$  manifolds then  $\varphi : \tilde{N} \rightarrow N \subset M$  is an immersion iff  $\varphi : \tilde{N} \rightarrow N$  is bijective,  $\varphi$  is smooth and also  $d\varphi : T_p \tilde{N} \rightarrow T_{\varphi(p)} M$  injective for all  $p$ .

### Example (Non-immersion)

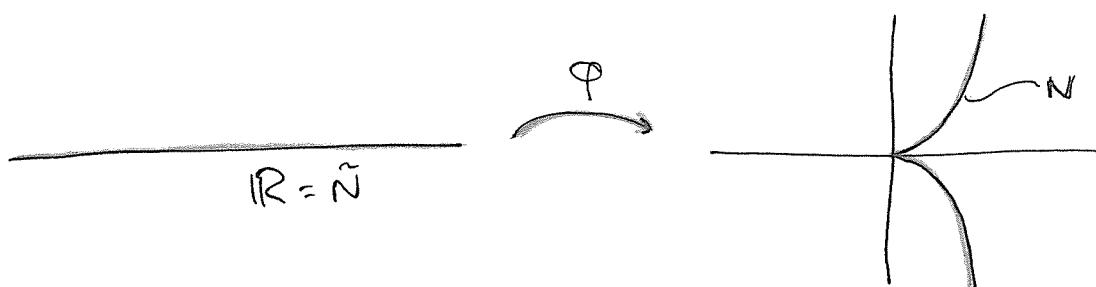
$$\varphi : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$\varphi(t) = (t^2, t^3)$$

$$\varphi(t_1) = \varphi(t_2) \Rightarrow t_1^3 = t_2^3 \Rightarrow t_1 = t_2 \text{ thus } \varphi \text{ is 1-1}$$

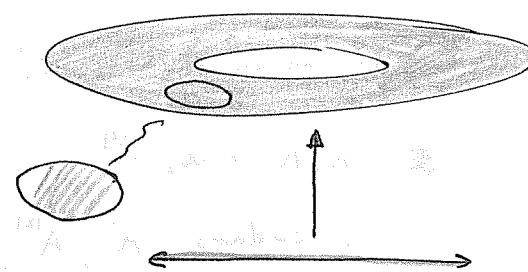
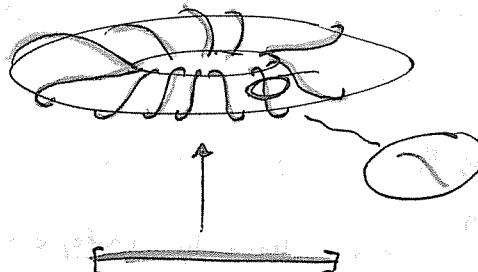
$$\text{But } \exists J_\varphi(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} \text{ and } \det(J_\varphi(t)) = 0 \text{ for } t=0$$

$$M = \mathbb{R}^2$$



(1.10)

Def<sup>b</sup> Let  $M \subseteq N$  is a regular submanifold iff  $\exists$  an immersion  $\varPhi : \tilde{N} \rightarrow N \subseteq M$  such that  $\forall p \in N$   $\exists$  arbitrarily small open sets  $U$  about  $p \in \varPhi^{-1}(U \cap N)$  is open and connected in  $\tilde{N}$



There seems to be 2 types of submanifolds - when we wrap densely

$\varPhi : \varPhi \rightarrow (e^{i\varphi}, e^{\alpha i\varphi})$  will be an immersion but if we look at a set of  $N$  we will find it is badly disconnected. Happily regular submanifolds are connected and they def<sup>b</sup> basically for that.

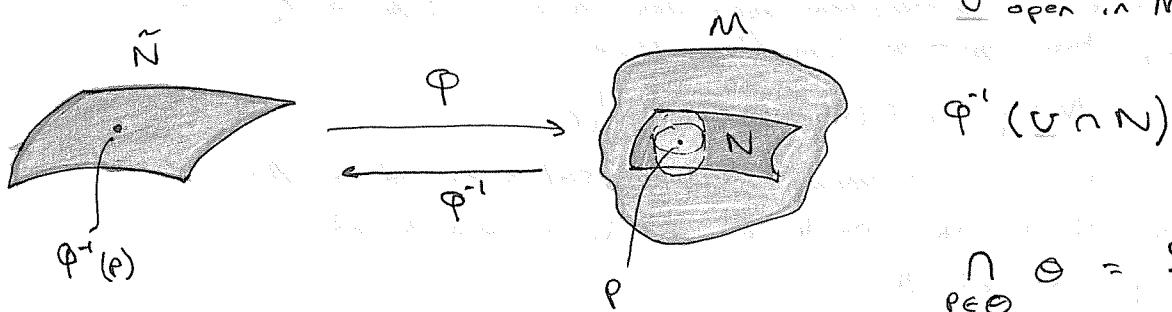
Lemma: An  $n$ -dimensional submanifold  $N \subseteq M$  is regular iff at each  $p \in N$   $\exists$  a chart  $(U, x)$  of  $M$  such that

$$N \cap U = \{q \mid x^{n+1}(q) = x^{n+2}(q) = \dots = x^m(q) = 0\}$$

$U \cap N$  then  $x_n(q) = (x^{n+1}(q), \dots, x^m(q))$

$(U \cap N, x_n)$  is a chart on  $N$

$$\varphi: \tilde{N} \rightarrow N \subseteq M$$



### Theorem

$$S \subseteq \mathbb{R}^3 \text{ where } S = f^{-1}(0)$$

If  $\exists f: \mathbb{R}^3 \rightarrow \mathbb{R}$  such that  $d_p f$  is invertible  $\forall p \in f^{-1}(0)$   
then  $f^{-1}(0)$  is a regular submanifold of  $\mathbb{R}^3$

injective

### Example

$$f(x, y, z) = 2x^2 + 4y^2 + z^2 - 1 \quad \frac{\partial f}{\partial z} = 4z \quad z = \sqrt{1-2x^2-4y^2}$$

$$S = \{(x, y, z) \mid f(x, y, z) = 0\} = f^{-1}(0) \quad \frac{\partial f}{\partial y} = 8y$$

$$\text{Pf} \quad d_p f(v) = v \cdot J_f(p)^t = v \cdot \nabla f(p)$$

$$d_p f \text{ surjective} \iff v \cdot \nabla f(p) \neq 0$$

$$\text{Discussion) } \nabla f(p_0) \neq 0 \Rightarrow \frac{\partial f}{\partial x}(p_0) \text{ or } \frac{\partial f}{\partial y}(p_0) \text{ or } \frac{\partial f}{\partial z}(p_0) \neq 0, \text{ at least one.}$$

wlog  $\frac{\partial f}{\partial z}(p_0) \neq 0$  then  $f(x, y, z) = 0 \Leftrightarrow z = g(x, y)$  in  $\text{nbhd}(p_0)$ .

Call  $\text{nbhd}(p_0) = U_0$  then consider  $U_0 \cap S \rightarrow \mathbb{R}^2$ ,  $(x, y, z) \xrightarrow{\varphi} (x, y)$

where  $\varphi^{-1}(x, y) = (x, y, g(x, y))$  so  $\varphi$  is a chart at  $p_0$ . Further

Suppose  $\frac{\partial f}{\partial y}(p_0) \neq 0$  and  $\exists V_0$  with  $f(x, y, z) = 0 \Leftrightarrow y = h(x, z)$

Then for  $V_0 \cap S \rightarrow \mathbb{R}^2$  we have  $\psi(x, y, z) = (x, z)$  and  $\varphi^{-1}(x, z) = (x, h(x, z), z)$ .

Consider then  $(\psi \circ \varphi^{-1})(x, y) = (x, g(x, y))$ .

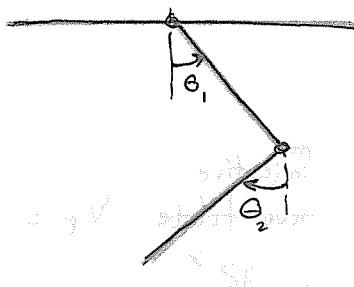
### Theorem 1.13

If  $M$  is an  $m$ -dimensional manifold and  $F : M \rightarrow \mathbb{R}^n$  is smooth, then  $m \geq n$  implies that

$$N = \{P \mid F(P) = 0\} = F^{-1}(0)$$

is an  $m-n$  dimensional regular submanifold of  $M$  provided that the rank of  $F$  is maximal at each point of  $N$ .

### Example



$$T^*Q = \{(q^1, \dots, q^n, p_1, \dots, p_n)\}$$

$$(\theta_1, \theta_2, p_1, p_2) \in T \times T \times \mathbb{R} \times \mathbb{R}$$

What is  $F^{-1}(0)$ ?

$$F(q) = (F_1(q), \dots, F_n(q))$$

$$F(q) = 0 \Leftrightarrow F_i(q) = 0 \quad \forall i$$

$$F_i : M \rightarrow \mathbb{R}$$

$$\bigcap_{i=1}^n F_i^{-1}(0) = F^{-1}(0)$$

$$F : M^n \rightarrow \mathbb{R}^7$$

$F_i^{-1}(0)$  =  $1^{\text{st}}$  dimensional submanifold  
 $F_2^{-1}(0)$  =  $1^{\text{st}}$  dimensional submanifold

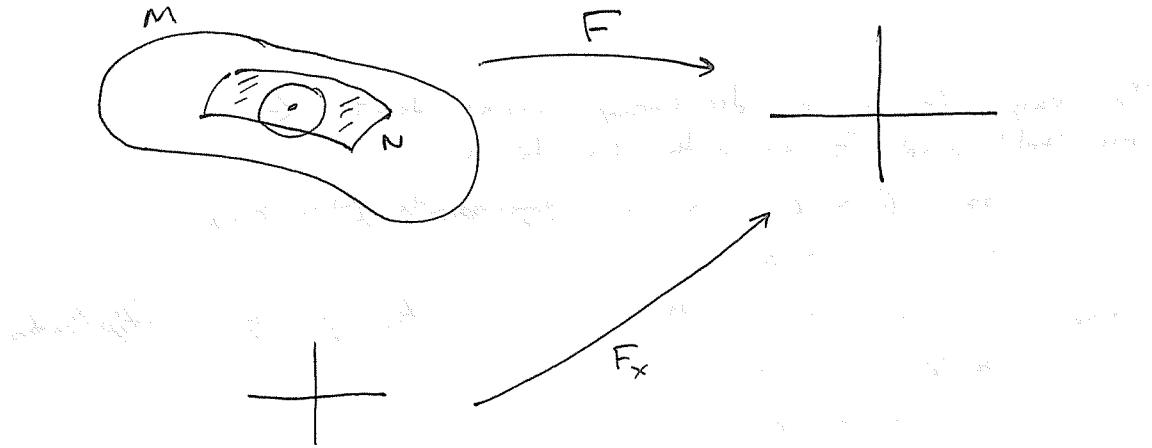
### Proof

At a point of  $N$ ,  $\text{rank } J_F = n$ , so  $\exists$  a chart  $(U, x)$  at the point such that  $F_x(u^1, u^2, \dots, u^m) = (u^1, u^2, \dots, u^n)$  where

$$F_x = \text{id} \circ F \circ x^{-1}$$

$$\text{For } q \in U \text{ and } F(q) = 0 \Leftrightarrow F_x(x(q)) = 0$$

$$\Leftrightarrow (x^1(q), x^2(q), \dots, x^n(q)) = 0$$



$$q \in U \cap N \Leftrightarrow x^i(q) = 0 \quad 1 \leq i \leq n$$

It follows that a chart on  $U \cap N$  is given by

$$q \rightarrow (x^{n+1}(q), \dots, x^m(q))$$

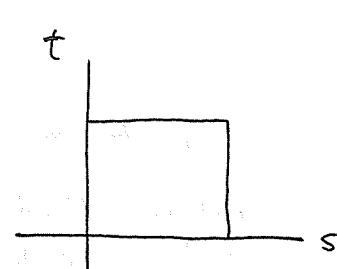
QED

Theorem

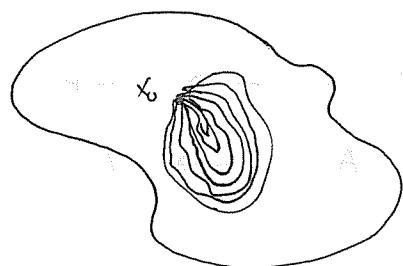
If  $M$  is a manifold then it is connected iff  $M$  is path connected.

Def<sup>n</sup>/ Path Connected :  $\forall p, q \in M \exists \alpha : [0, 1] \rightarrow M$  such that  $\alpha(0) = p$  and  $\alpha(1) = q$

Def<sup>n</sup>/ Simply Connected : Can continuously deform loop to a point



$$h(s, t) = \begin{cases} \alpha(s) & t=0 \\ x_0 & t=1 \end{cases}$$



## LIE Groups

Def<sup>b</sup>/ To say  $G$  is a Lie Group means that  $G$  is a manifold and  $\exists$  smooth functions

$$m: G \times G \rightarrow G \quad m(g, h) = g \cdot h$$

$$\text{ir}: G \rightarrow G ;$$

and if  $a, b \in G$  then  $m$  is the group multiplication

$$m(a, b) = a \cdot b$$

$$\text{ir}(a) = a^{-1}$$

### Example

$$\{(e^{i\theta}, e^{i\phi}) \mid \theta, \phi \in \mathbb{R}\}$$

$$S' = \{e^{i\theta} \mid \theta \in \mathbb{R}\} \quad e^{i\theta} e^{i\phi} = e^{i(\theta + \phi)}$$

$$\mu(e^{i\theta}) = \theta \quad 0 \leq \theta < 2\pi$$

$$\delta(e^{i\theta}) = \theta \quad -\pi < \theta < \pi$$

$$m(e^{i\theta}, e^{i\phi}) = e^{i\theta} e^{i\phi}$$

### Example

$$\text{GL}(n) = \{A \mid \det(A) \neq 0\}$$

$$\mathbb{R}^{n^2} \cong \text{gl}(n) \longrightarrow \mathbb{R}$$

$$f(A) = \det(A) \leftarrow \text{polynomial}$$

$$\text{GL}(n) = \text{gl}(n) \setminus f^{-1}(0)$$

$$(A, B) \longrightarrow AB \quad \text{smooth because polynomial}$$

$$A \longrightarrow A^{-1} = \frac{1}{\det(A)} A^{\text{adj}} \leftarrow \text{rational function with no poles}$$

$\therefore \text{GL}(n)$  is a Lie group.

Topic : Show  $\text{SO}(n)$  is a Lie Group

$$f : \text{gl}(n) \rightarrow \text{gl}(n)$$

$$Df(A) : \text{gl}(n) \rightarrow \text{gl}(n)$$

$J_f(A)$  is an  $n^2 \times n^2$  matrix  $\leftarrow$  Not easy to check rank

$\Rightarrow$  We instead proceed in non-standard way consisting of ~~at least~~ 6 steps

① Let  $\mathcal{S}_n = \mathcal{S}(n) =$  set of all  $n \times n$  symmetric matrices

Then  $\mathcal{S}(n)$  is a subspace of  $\text{gl}(n)$  of dimension  $\frac{n^2+n}{2}$ .

Pf: Let  $E_j^i$  denote matrix  $e^i \cdot (e^j)^T$ .  $A \in \mathcal{S}_n \Rightarrow A^T = A$

$$\Rightarrow A = A^T$$

$$\Rightarrow A = \frac{1}{2}(A + A^T)$$

$$\Rightarrow A = \frac{1}{2}(A_i^j E_j^i + (A_i^j E_j^i)^T)$$

$$\Rightarrow A = \sum_{i,j} A_i^j \left[ \frac{1}{2}[E_j^i + (E_j^i)^T] \right]$$

$$\Rightarrow A = \sum_{i,j} A_i^j \tilde{E}_j^i ; \quad \tilde{E}_j^i = \frac{1}{2}(E_j^i + E_i^j)$$

And this  $\tilde{E}_j^i$  serves as a basis for  $\mathcal{S}(n)$ .

$$\frac{n^2-n}{2} + n = \frac{n^2+n}{2} \therefore \dim \mathcal{S}(n) = \frac{n^2+n}{2}$$

②  $\forall A \in \text{gl}(n), A^T A^T - I \in \mathcal{S}(n)$

$$(AA^T - I)^T = (AA^T)^T - I^T = A^T A^T - I = AA^T - I$$

③ We are trying to get a hold of the image of  $Df(A)$   
(which is isomorphic to  $\mathcal{S}(n)$  actually)

Let  $f : \text{gl}(n) \rightarrow \mathcal{S}(n)$  defined by  $f(A) = AA^T - I$

We shall show  $H \in \text{Ker}(Df(A)) \Leftrightarrow AH^T \in A\mathcal{S}(n)$

$$f^{-1}(0) = \{A \mid f(A) = 0\} = \{A \mid AA^T = I\} = \text{SO}(n)$$

Let  $A\mathcal{S}(n) = \text{antisymmetric mat's} = \{A \in \text{gl}(n) / A = -A^T\}$

$\mathbb{F}^{n \times n}$  on  $\mathbb{R}^{n \times n}$

③ We have to show  $H \in \text{Ker } Df(A) \iff AH^T \in \mathcal{A}\mathcal{S}(n) = \text{Antisymmetric } n \times n$ 's

$$Df(A)(H) = AH^T + HA^T$$

$$\begin{aligned}\text{Ker } Df(A) &= \{H \mid Df(A)(H) = 0\} = \{H \mid AH^T + HA^T = 0\} \\ &= \{H \mid AH^T = -HA^T = -(AH^T)^T\}\end{aligned}$$

④ Let  $A \in \mathcal{M}(n)$   $\neq f(A) = 0$  and let  $K_A = \text{Ker } (Df(A))$

Let  $L_A : K_A \rightarrow \mathfrak{gl}(n)$  be defined by  $L_A(H) = AH^T$

We show  $\text{Ker } (L_A) = \{0\}$  and so  $L_A$  is 1-1.

/Pf/ If  $L_A(H) = 0$  then  $AH^T = 0 \therefore A^T A H^T = A^T 0$   
 $\therefore H^T = 0 \therefore \underline{H = 0}$

Further more we may draw the conclusions,

$$\text{Ker } (Df(A)) = \{H \mid L_A(H) \text{ is skew}\}$$

and  $L_A$  is one-one

$$\therefore L_A(\text{Ker } (Df(A))) = \text{set of skew} = \mathcal{A}\mathcal{S}(n) \text{ matrices}$$

$\therefore K_A = \text{Skew Symmetric matrices.}$

$$K_A = L_A^{-1}(\mathcal{A}\mathcal{S}(n))$$

$K_A \xrightarrow{L_A \text{ is } 1-1 \text{ and onto}} \mathcal{A}\mathcal{S}(n); L_A$  1-1 and onto

$$\therefore \dim(\mathcal{A}\mathcal{S}(n)) = \frac{n^2-n}{2} = \boxed{\dim K_A = \frac{n^2-n}{2}}$$

$AH^T = (A^T)^T H^T$  (if  $A$  is antisymmetric  $\Rightarrow A^T = -A$ )

Now  $A^T = -A$  if and only if  $A^T A = 0$  (if and only if  $A^T A = 0$ )

⑤ If  $A \in f^{-1}(0)$  then  $\text{rank } Df(A) = \frac{n^2+n}{2} = \dim \mathcal{S}(n)$

$$f: \mathfrak{gl}(n) \rightarrow \mathcal{S}(n)$$

$$Df(A): \mathfrak{gl}(n) \rightarrow \mathcal{S}(n)$$

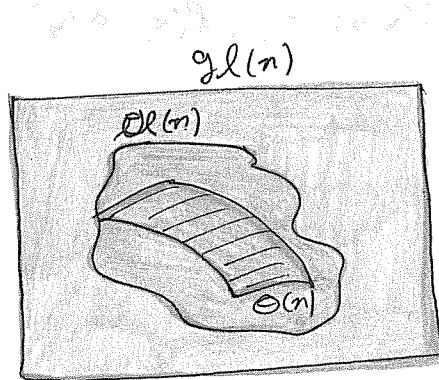
$$\dim(\mathfrak{gl}(n)) = n^2$$

$$\begin{aligned} n^2 &= \dim \text{Nullspace}(Df(A)) + \text{rk}(Df(A)) \\ &= \frac{n^2-n}{2} + \text{rank}(Df(A)) \end{aligned}$$

$\Rightarrow Df(A)$  has rank  $\frac{n^2+n}{2}$

$\Rightarrow Df(A)$  has maximal rank at each  $A \in f^{-1}(0)$ .

⑥  $\therefore \mathcal{O}(n) = f^{-1}(0)$  is a regular submanifold of  $\mathfrak{gl}(n)$  and also of  $\mathcal{M}(n)$  by theorem 7.13



Homework) Lorentz Matrices =  $\{A \in \mathfrak{gl}(4) \mid A \cdot n \cdot A^T = n\}$

$$n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Show that Lorentz Matrices

Actually form a Lie Group.

⑥ Show  $\Theta(n)$  is a Lie group, we have that  $\Theta(n)$  is regular sub-manifold if still remains for us to show that the group operations are smooth,

$$i : \Theta(n) \hookrightarrow \mathcal{G}l(n)$$

$$i(x) = x$$

$$m : \mathcal{G}l(n) \rightarrow \mathcal{G}l(n)$$

$$m(A, B) = AB$$

$$\Theta(n) \times \Theta(n) \longrightarrow \Theta(n)$$

$i \circ m \circ (i \times i)$  is the multiplication on  $\Theta(n)$  and is smooth.

Notice] If we have

$$f : M \rightarrow N$$

$$g : P \rightarrow Q$$

$$(f \times g)(x, y) = (f(x), g(y))$$

$(U, \alpha)$  chart on  $M$

$(V, \beta)$  chart on  $N$

$(\mathbb{W}, \gamma)$  chart on  $P$

$(X, \omega)$  chart on  $Q$

$$(U \times V) \rightarrow \alpha(U) \times \beta(V)$$

$$(\mathbb{W} \times X, \gamma \times \omega) \longrightarrow \gamma(W) \times \omega(X)$$

$$(\gamma \times \omega) \circ (f \times g) \circ (\alpha \times \beta)^{-1} = (\gamma \circ f \circ \alpha^{-1}) \circ (\omega \circ g \circ \beta^{-1})$$

Warner

$\dot{z} \circ iv \circ z \Rightarrow iv$  smooth operation

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## Product Manifolds

1/31/2021

If  $M, N$  are manifolds then there is a natural manifold structure on  $M \times N$ .

$a_M = \text{atlas for } M$

$a_N = \text{atlas for } N$

$(U \times V, \times_{xy})$  is a chart on  $M \times N$  where

$$U \times V \xrightarrow{\times_{xy}} X(U) \times Y(V) \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$$

$$(\times_{xy})(u, v) = (x(u), y(v))$$

This is the manner in which we construct  $a_{M \times N}$

$$a_{M \times N} = \left\{ (U \times V, \times_{xy}) \mid \begin{array}{l} (U, x) \in a_M \\ (V, y) \in a_N \end{array} \right\}$$

## Submanifolds of Products

Fix  $m_0 \in M$  then the map  $N \xrightarrow{\phi} M \times N$  defined by

$$\phi(y) = (m_0, y)$$

is a submanifold map. Likewise fix  $n_0 \in N$  the map

$$\psi(x) = (x, n_0)$$

is also a submanifold map.

## Proposition

If  $G$  and  $H$  are Lie Groups so is  $G \times H$ . Also  $G, H$  are regular submanifolds of  $G \times H$  which are also subgroups

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

$$(g, h)^{-1} = (g^{-1}, h^{-1})$$

$$m: (G \times H) \times (G \times H) \longrightarrow G \times H$$

Poincaire  
 $\mathbb{R}^4 \oplus \mathcal{L}$

$$\begin{aligned} m((g_1, h_1) \circ (g_2, h_2)) &= (g_1 g_2, h_1 h_2) \\ &= (m_G(g_1, g_2), m_H(h_1, h_2)) \end{aligned}$$

$$\begin{aligned} S^1 &= \{e^{i\theta} \mid \theta \in \mathbb{R}\} \subseteq \mathbb{C} \\ &= \{\cos \theta + i \sin \theta \mid \theta \in \mathbb{R}\} \end{aligned}$$

$$\longleftrightarrow \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Def<sup>h</sup>/ If  $G$  and  $H$  are Lie Groups a Lie group homomorphism is a smooth map  $\phi: G \rightarrow H$  which is also a homomorphism.

Def<sup>a</sup>/ A Lie homomorphism is a Lie isomorphism iff  $\phi^{-1}$  exists and is smooth from  $H \rightarrow G$ .

Def<sup>n</sup>/ A Lie subgroup  $H \subseteq G$ ,  $G$  a Lie group, is defined by an immersion  $\Phi: \tilde{H} \rightarrow H \subseteq G$  where  $\tilde{H}$  is a Lie group and  $\Phi$  is a Lie homomorphism from  $\tilde{H}$  to  $G$ .

So  $\exists$  ~~submanifolds~~ that are not Lie Groups but if we add a little structure then submanifold  $\Rightarrow$  Lie subgroup

$$\begin{aligned} &\text{A Lie group } G = (M, \cdot, \cdot^{-1}, \cdot^{-1}) \\ &\text{A submanifold } H \subseteq M \end{aligned}$$

$\mathcal{G}(Ax, Ay)$

THEOREM

Assume  $G$  is a Lie group and that  $G \ni H$  is an immersed Lie subgroup of  $G$ ,  $\varphi: \tilde{H} \rightarrow H \subseteq G$ . Then  $H \subseteq G$  is closed iff  $H$  is a regular submanifold of  $G$ .

NON TRIVIAL  
THEOREM

Corollary

If  $G$  is a Lie group and  $H \subseteq G$  is an algebraic subgroup such that  $H = F^{-1}(0)$  for some continuous function  $F: G \rightarrow \mathbb{R}^m$  then  $H$  is a regular Lie subgroup of  $G$ .

$$\varphi: \tilde{H} \rightarrow H$$

$$m_H = \varphi^{-1} \circ m_G \circ (\varphi \times \varphi)$$

$$iv_H = \varphi^{-1} \circ iv_G \circ \varphi$$

$$SO(n) = \{A \mid A^T A = I, \det(A) = 1\}$$

$$\det: \Theta(n) \rightarrow \mathbb{R}$$

$\det: GL(n) \rightarrow \mathbb{R}$  is smooth because  $\det$  is just a polynomial in  $A \in GL(n)$  and polynomials are smooth.

$$F: GL(n) \rightarrow gl(n) \times \mathbb{R}, \text{ again } F \text{ just poly} \therefore \text{smooth}$$

$$F(A) = (A^T A - I, \det(A) - 1)$$

All we need is that  $F$  is continuous which it is so then

$$F^{-1}(0,0) = \{A \mid A^T A = I, \det(A) = 1\} = SO(n)$$

$\odot(n)$  contains rotations and reflections

### Corollary 2

Assume  $G$  is a Lie group and  $H$  as well, and that  $\phi: G \rightarrow H$  is a surjective Lie homomorphism then the kernel( $\phi$ ) is a Lie subgroup and  $G/\ker\phi \cong H$  Lie isomorphism

$$\text{Notice, } \text{Kernel } \phi = \phi^{-1}(\{e_H\}) = \{x \in G \mid \phi(x) = e_H\}$$

Think of the densely wound torus again. Our principle counterexample.

$$\phi: \mathbb{R} \rightarrow T$$

$$\phi(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$$

$$\alpha = \frac{p}{q}, p, q \in \mathbb{Z}, \gcd(p, q) = 1.$$

$$\phi(q) = (e^{2\pi i q}, e^{2\pi i \frac{p}{q}q}) = (1, 1)$$

$$\ker \phi \supseteq q\mathbb{Z} \rightarrow q\mathbb{Z} \subseteq \ker \phi$$

$$\phi(t) = (1, 1)$$

$$e^{2\pi i t} = 1$$

$$e^{2\pi i \alpha t} = 1$$

$$\therefore \alpha t \in \mathbb{Z} \text{ but } \frac{p}{q}t = n = \alpha t$$

$$pt = nq \therefore p/n \therefore p = pr \text{ for } r \in \mathbb{Z}$$

$$\text{Thus } pt = prq \Rightarrow t = rq \therefore t \in q\mathbb{Z}.$$

$$S_1 \xrightarrow[q\mathbb{Z}]{} \mathbb{R} \cong T \leftarrow \text{torus}$$

Now if  $\alpha$  is irrational then  $\left\{ \begin{array}{l} e^{2\pi i t} = 1 \\ e^{2\pi i \alpha t} = 1 \end{array} \right\}$   
has only the  $t=0$  sol<sup>h</sup>  
So  $\text{Pois}(t)$  Kernel is just zero.

$GL(n) \subseteq gl(n)$

↑ open dense subset of  $gl(n)$

Next Time : Local Lie Groups and Actions



Lie : Did everything with local Lie Groups

- Was proven that global Lie Groups make sense topologically...
- Unique simply connected...
- Marsden all about some local Lie Groups

### Local Lie Group

A Local Lie Group consists of open connected sets  $V_0 \subseteq V \subseteq \mathbb{R}^n$

and smooth maps  $m: V \times V \rightarrow \mathbb{R}^n$  and  $i: V_0 \rightarrow V$  such that

1.)  $\forall g_1, g_2, g_3 \in V$  such that  $m(g_1, g_2) \in V$  and  $m(g_2, g_3) \in V$

and  $(m(g_1, g_2), g_3) \in V \times V$ ,  $(g_1, m(g_2, g_3)) \in V \times V$  (redundant line)

then  $m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3))$  Associativity of  $m$

where the question of Associativity makes sense.

2.)  $0 \in V_0$  that is  $m(g, 0) = g = m(0, g) \quad \forall g \in V$

3.)  $\forall g \in V_0$  we have  $m(g, i(g)) = 0 = m(i(g), g)$

Basically a Local Lie Group is an open connected subset with a smooth map  $m$  which represents the group operation where that makes sense.

### Example

$$(\mathbb{R}^n, +), \quad U \subseteq \mathbb{R}^n, \quad U = (-1, 1) \subseteq \mathbb{R}$$

$$U^2 = \{a \cdot b \mid a \in U, b \in U\} \quad \frac{3}{4} \in U, \quad \frac{3}{4} + \frac{3}{4} \notin U$$

$$\partial U = \{a + b \mid a \in U, b \in U\}$$

GL(n) ? How to make  $m$ ?

Perhaps  $m(A, B) = (I - A)(I - B)$   
 $m(0, B) = I(I - B)$ , nope

Hmm... how to make  
 $GL(n)$  a local Lie Group  
if it is a Global Lie Group

a Lie Group  $G$  acts locally on a manifold  $M$  iff  
 $\exists U \subseteq G \times M$  open and a <sup>smooth</sup> mapping  $\psi: U \rightarrow M$  such that

- 1.) if  $(h, x) \in U$  and  $(g, \psi(h, x)) \in U$  and  $(gh, x) \in U$   
 then  $\psi(g, \psi(h, x)) = \psi(gh, x) \Leftrightarrow g \cdot (h \cdot x) = (gh) \cdot x$
- 2.)  $\forall x \in M, (e, x) \in U$  and  $\psi(e, x) = x \Leftrightarrow e \cdot x = x$

This is a Lie Group acting locally.

To say that a Lie Group  $G$  acts on a Manifold  $M$  means  
 that  $\exists$  smooth functions  $\psi: G \times M \rightarrow M$  such that if

$$g \cdot x = \psi(g, x) \quad \forall (g, x) \in G \times M$$

Then we have that  $\forall x \in M$  and  $\forall g, h \in G$ .

$$g \cdot (h \cdot x) = (gh) \cdot x \quad (\text{because } \psi \text{ is } (G \times M) \rightarrow M)$$

$$e \cdot x = x \quad (\text{because } \psi \text{ is } (G \times M) \rightarrow M)$$

### Trivial Example

$$\mathrm{GL}(n) \times V \rightarrow V$$

$$(A, v) \rightarrow v$$

$$\psi(A, v) = v \quad \forall A \in \mathrm{GL}(n), \forall v \in V \quad (\text{because } \psi \circ \psi = \psi)$$

$$(AB) \cdot v = v = A \cdot (B \cdot v)$$

$$e \cdot v = v$$

Another Mapping is easy to get as

$$\hat{\psi}: G \rightarrow \mathrm{Diff}(M) \quad \text{defined by} \quad \hat{\psi}(g)(x) = \psi(g, x) \quad (\text{because } \psi \text{ is } (G \times M) \rightarrow M)$$

$\uparrow$  Group (as dimensional)

$$\hat{\psi}(A): V \rightarrow V \quad \text{where} \quad \hat{\psi}(A)(v) = \psi(A, v) = v$$

$$\hat{\psi}(A)(v) = v \quad \forall v \quad \therefore \hat{\psi}(A) = \mathrm{id}_V \quad (\text{because } \psi \text{ is } (G \times M) \rightarrow M)$$

So,  $\hat{\psi}: \mathrm{GL}(n) \rightarrow \mathrm{GL}(V)$  if  $\psi(\cdot, \cdot)$  is linear

$$\mathrm{GL}(V) = \{ L \mid L: V \rightarrow V, \text{ $f^{-1}$ exists and } L \text{ linear} \}$$

Example 2 : THE STANDARD REPRESENTATION

2/5/2001

$\psi: \mathcal{G}\ell(n) \times V \rightarrow V$  where  $V = \mathbb{R}^n$  and  $\psi(A, x) = Ax$

$\hat{\psi}(A)(x) = Ax \quad \forall x \Rightarrow \hat{\psi}(A) = A \Rightarrow \hat{\psi} = \text{id}_{\mathcal{G}\ell(n)}$

Assumed to  $\hat{\psi}(A) = \text{id}_V$

Example 3 :

~~$\psi: \mathcal{G}\ell(n) \times V \rightarrow V$~~

$\psi: \mathcal{G}\ell(n) \times V \rightarrow V \quad \text{fix } k \in \mathbb{N}$

$$\psi(A, v) = \det(A)^k v$$

$$\begin{aligned} \psi(AB, v) &= (AB)v = \det(AB)v = (\det A)^k(\det B)^k v = \psi(A, \psi(B, v)) \\ \psi(I, v) &= v \end{aligned}$$

Example 4

$\psi: \mathcal{G}\ell(n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  where  $\psi(A, x) = (A^{-1})^t x$

Example 5

a)  $\psi: \mathcal{G}\ell(n) \times \mathcal{G}\ell(n) \rightarrow \mathcal{G}\ell(n)$  where  $\psi(A, B) = ABA^{-1}$

b)  $\psi: \mathcal{G}\ell(n) \times \mathcal{G}\ell(n) \rightarrow \mathcal{G}\ell(n)$  where  $\psi(A, B) = ABA^t$

In the case of b) you get other actions by restriction as follows

$\tilde{\psi}: \mathcal{G}\ell(n) \times \mathcal{S}(n) \rightarrow \mathcal{S}(n) \text{ by } \psi(A, B) = ABA^t$

$\tilde{\psi}_A: \mathcal{G}\ell(n) \times \mathcal{AS}(n) \rightarrow \mathcal{AS}(n) \text{ by } \psi(A, B) = ABA^t$

### Example 6

$$\psi: \mathrm{GL}(n) \times (\mathbb{R}^n)^* \longrightarrow (\mathbb{R}^n)^*$$

Where  $(\mathbb{R}^n)^*$  = vector space of all linear maps from  $\mathbb{R}^n \rightarrow \mathbb{R}$

$$\alpha: \mathbb{R}^n \rightarrow \mathbb{R} \text{ then } \alpha(x) = a \cdot x = a_1 x^1 + a_2 x^2 + \dots + a_n x^n$$

Where we define action of  $\psi$  by

$$\psi(A, \alpha) = A \cdot \alpha = \bar{\alpha}$$

$$(A \cdot \alpha): \mathbb{R}^n \rightarrow \mathbb{R}$$

$$(A \cdot \alpha)(x) = \alpha(A^{-1}x) \in \mathbb{R} \leftarrow \text{covariant vector}$$

For Example

$$\alpha: \mathbb{R}^3 \rightarrow \mathbb{R}, \alpha(x, y, z) = x + y, A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

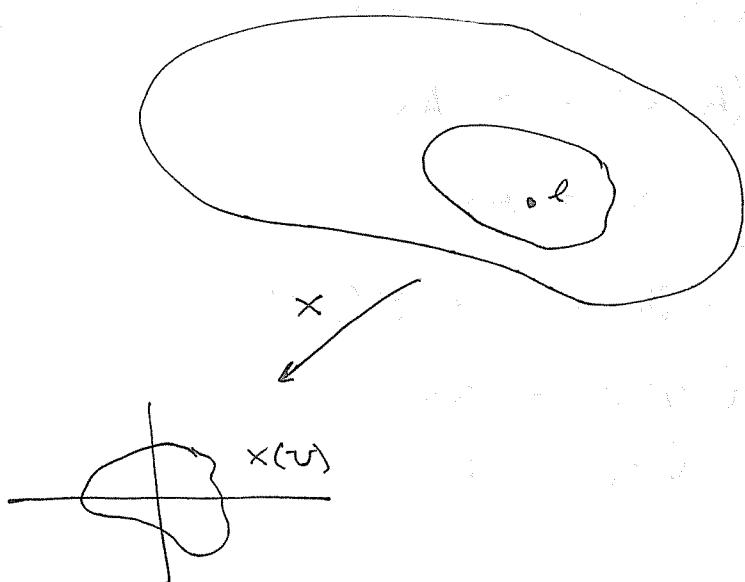
$$(A \cdot \alpha)(x, y, z) = \alpha(A^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix})$$

$$\begin{aligned} [(AB) \cdot \alpha](x) &= \alpha((AB)^{-1}x) = \alpha(B^{-1}(A^{-1}x)) = (B \circ \alpha)(A^{-1}x) \\ &= [A \circ (B \circ \alpha)](x) \quad \forall x \end{aligned}$$

$$\therefore A[(AB) \cdot \alpha] = [A \circ (B \circ \alpha)]$$

## G Lie group

$(v, x)$  a chart &  
modify chart so that  
 $x(\lambda) = 0$ . Make a  
open connected subset  
of  $x(v)$  into local Lie  
Group



$$\tilde{x}(p) = x(p) - x(\lambda)$$

$$\tilde{x} = \varphi \circ x$$

$$\tilde{x} \circ \tilde{x}^{-1} = \varphi$$

$\varphi$  diffeomorphism

So then we can construct

$$m : (x(v) \times x(v)) \rightarrow x(v)$$

$$m(u, v) = x(x^{-1}(u) x^{-1}(v))$$

But can we take  $x^{-1}(u) x^{-1}(v)$  into  $x$ ? well if we restrict the domain then we can and that works since the operation is continuous

---


$$\text{SL}(n) \times$$

REPRESENTATIONS

$$\boxed{\mathcal{GL}(n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n}$$

$$(A, x) \rightarrow Ax$$

$$\psi(A, x) = Ax$$

$$\hat{\psi} : \mathcal{GL}(n) \rightarrow \mathcal{GL}(\mathbb{R}^n)$$

$$\hat{\psi}(A)x = Ax$$

$$\hat{\psi}(A) = A$$

$$\mathcal{GL}(n) \times (\mathbb{R}^n)^* \xrightarrow{\psi} (\mathbb{R}^n)^*$$

$$\psi : (A, \alpha) \xrightarrow{\psi} A \cdot \alpha = \beta$$

$$\beta(x) = (A \cdot \alpha)(x) = \alpha(A^{-1}x)$$

$$\hat{\psi}(A)(\alpha) = A \cdot \alpha = \beta$$

$$\hat{\psi}(A) : (\mathbb{R}^n)^* \rightarrow (\mathbb{R}^n)^*$$

$$\hat{\psi}(A) \in \mathcal{GL}((\mathbb{R}^n)^*)$$

$$\hat{\psi} : \mathcal{GL}(n) \rightarrow \mathcal{GL}((\mathbb{R}^n)^*)$$

$G \times V \xrightarrow{\psi} V$  //  $G$  an abstract Group, Assumption linear representation

$$\psi(g, c_1 v_1 + c_2 v_2) = c_1 \psi(g, v_1) + c_2 \psi(g, v_2)$$

$\hat{\psi}$  is a group homomorphism

$$\hat{\psi} : G \rightarrow \mathcal{GL}(V)$$

$$\begin{aligned}\hat{\psi}(g_1 g_2)(v) &= \psi(g_1 g_2, v) \\ &= \psi(g_1, \psi(g_2, v)) \\ &= \psi(g_1, \hat{\psi}(g_2)(v)) \\ &= \hat{\psi}(g_1)(\hat{\psi}(g_2)(v)) \\ &= [\hat{\psi}(g_1) \circ \hat{\psi}(g_2)](v)\end{aligned}$$

$$\hat{\psi}(g)(x) = \psi(g, x) \in V$$

$$\hat{\psi}(g) : V \rightarrow V$$

$$\hat{\psi}(g) \in \mathcal{GL}(V)$$

$$\hat{\psi}(g_1 g_2)(v) = \hat{\psi}(g_1)(\hat{\psi}(g_2)(v))$$

$$\hat{\psi}(g_1 g_2) = \hat{\psi}(g_1) \circ \hat{\psi}(g_2) \quad \leftarrow \text{Modules \& Representations Connection}$$

A representation is a homomorphism typically from an abstract group to something more familiar.

### TENSORS

If  $\alpha, \beta \in (\mathbb{R}^n)^*$  define  $\alpha \otimes \beta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$

$$(\alpha \otimes \beta)(x, y) = \alpha(x)\beta(y)$$

Now  $\alpha \otimes \beta$  is a bilinear map from  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ .

$$(\alpha \otimes \beta)(c_1 x_1 + c_2 x_2, y) = c_1 \alpha \otimes \beta(x_1, y) + c_2 \alpha \otimes \beta(x_2, y)$$

Let  $\{e_1, e_2, \dots, e_n\}$  be a basis for  $\mathbb{R}^n$ . The dual basis of  $(\mathbb{R}^n)^*$  is  $\{e^1, e^2, \dots, e^n\}$  where

$$e^i : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$e^i(x) = x^i$$

$$e^i(e_j) = \delta_j^i$$

So then we see that,

$$(e^i \otimes e^j)(x, y) = e^i(x) e^j(y) = x^i y^j$$

$$\alpha = \sum_i \alpha_i e^i$$

$$\alpha(e_j) = \sum_i \alpha_i e^i(e_j) = \alpha_j$$

$$\beta = \sum_j \beta_j e^j$$

$$\alpha \otimes \beta = \left( \sum_i \alpha_i e^i \right) \otimes \left( \sum_j \beta_j e^j \right) = \sum_i \sum_j \alpha_i \beta_j (e^i \otimes e^j)$$

$\therefore \{e^i \otimes e^j\}$  span  $\{\alpha \otimes \beta \text{ type objects}\}$

THEOREM

$\{e^i \otimes e^j\}$  is a basis for the vector space of all bilinear maps from  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$

$$T = \sum_{i,j} T_{ij} (e^i \otimes e^j)$$

$$T : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

$$T(x, y) = T(\sum x^i e_i, \sum y^j e_j)$$

Notice that  $x = (x^1, x^2, x^3) = x^1(1, 0, 0) + x^2(0, 1, 0) + x^3(0, 0, 1)$

$$T(x, y) = \sum_i \sum_j x^i y^j T(e_i, e_j)$$

$$= \sum_i \sum_j T_{ij} x^i y^j \quad \leftarrow \text{polynomial in vector variables}$$

$$(e^i \otimes e^j)(x, y) = x^i y^j \quad \text{from last page}$$

$$\therefore T(x, y) = \sum_i \sum_j T_{ij} (e^i \otimes e^j)(x, y)$$

How about the group action

$$\mathrm{SL}(n) \times T_2 \mathbb{R}^n \xrightarrow{\psi} T_2 \mathbb{R}^n$$

$$\psi(A, T) = \sum_{i,j} T_{ij} (A \cdot e^i) \otimes (A \cdot e^j)$$

$$(A \cdot e^k)(x) = e^k (A^{-1} x) \quad \text{now } x = \sum x^i e_i : x \in \mathbb{R}^n$$

$$= e^k (\sum x^i (A^{-1} \cdot e_i))$$

$$= e^k \sum_i x^i e^k \sum_j (A^{-1})_i^j e_j$$

$$\begin{aligned}
 (A \circ e^\kappa)(x) &= \sum_i x^i \sum_j (A^{-1})_{ij}^j e^\kappa(e_j) \\
 &= \sum_{i,j} x^i (A^{-1})_{ij}^j \delta_j^\kappa \\
 &= \sum_i x^i (A^{-1})_{ii}^\kappa \\
 &= \sum_i (A^{-1})_{ii}^\kappa e^i(x)
 \end{aligned}$$

$$A \circ e^\kappa = \sum_i (A^{-1})_{ii}^\kappa e^i$$

So we see how a matrix acts on a dual vector (basis actually)

Now we can find explicit formula for how A acts on T

$$\psi(A, T) = \sum_{i,j} T_{ij} \left[ \left( \sum_k (A^{-1})_{ik}^j e^\kappa \right) \otimes \left( \sum_\ell (A^{-1})_{\ell j}^i e^\ell \right) \right]$$

$$\psi(A, T) = \sum_{i,j} \sum_{k,\ell} T_{ij} (A^{-1})_{ik}^j (A^{-1})_{\ell j}^i (e^\kappa \otimes e^\ell) : \text{Math}$$

$$A \circ (T_{ij}) = \left( T_{\kappa \ell} (A^{-1})_{ij}^\kappa (A^{-1})_{ij}^\ell \right) : \text{Physics}$$

$$T_{ij} \longrightarrow e_1, e_2, e_3, \dots, e_n$$

$$Ae_i = f_i$$

$$\bar{T}_{ij} \longrightarrow f_1, f_2, \dots, f_n$$

$$\hat{\psi} : \mathcal{G}\mathcal{L}(n) \longrightarrow \mathcal{G}\mathcal{L}(T^*_{\mathbb{R}^n})$$

Irreducible orthochronal representations of the Lorentz Group

↳ Built from tensors



①

$$x \in \mathbb{R}^n$$

$$\hat{x} : (\mathbb{R}^n)^* \rightarrow \mathbb{R} \Rightarrow \hat{x} \in (\mathbb{R}^n)^{**}$$

$$\begin{cases} (f+g)(x) = f(x) + g(x) \\ (cf)(x) = c f(x) \end{cases} \quad \hat{x}(\alpha) = \underline{\alpha}(x) \leftarrow \text{linear}$$

$$\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\underline{\alpha}(x)$$

$$(\hat{x} \otimes p)(x, y) = \underline{\alpha}(x)p(y)$$

$$|q\rangle \rightarrow \langle p | q \rangle$$

$$\langle p | \rightarrow \langle p | q \rangle$$

$$\psi : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^{**}$$

$$\underline{\psi(x)} = \hat{x}$$

$$\psi(x) = \hat{x}$$

$$\psi(y) = \hat{y}$$

$$\psi(x+y) = \hat{x+y} = \hat{x} + \hat{y} = \psi(x) + \psi(y)$$

$$\begin{aligned} \hat{x+y}(x) &= \alpha(x+y) = \alpha(x) + \alpha(y) = \hat{x}(x) + \hat{y}(x) \\ \psi(cx) &= c \psi(x) \end{aligned}$$

$$= (\hat{x} + \hat{y})(x)$$

$\psi$  is one-one + onto

$$\mathbb{R}^n \cong (\mathbb{R}^n)^{**}$$

$$\hat{x}(\alpha) = \underline{\alpha}(x)$$

$$\circledast \quad (\hat{x} \otimes \hat{y}) : (\mathbb{R}^n)^* \times (\mathbb{R}^n)^* \rightarrow \mathbb{R}$$

$$(\hat{x} \otimes \hat{y})(\alpha, \beta) = \hat{x}(\alpha) \hat{y}(\beta) = \underline{\alpha}(x) \underline{\beta}(y)$$

# Math 121 Quiz I Summer 1999

May 19, 1999

1. Simplify:

$$(2x^2y^{-4})^5.$$

2. Solve:

$$2 - 10x \leq 8 - 6x.$$

3. Find the slope of the line which contains the points  $(-1, 2)$  and  $(3, -1)$ .

4. Let  $f$  be the function defined by  $f(x) = x^2 - 3$ . Find

$$f(-1), f(0), f(3), f(x+h).$$

②

$\{e_i\}$  basis of  $\mathbb{R}^n$        $\{\hat{e}_i\}$

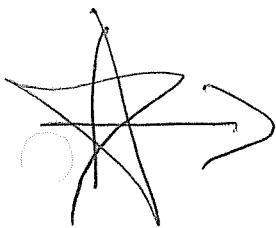
$$\hat{e}_i(\alpha) = \alpha(e_i)$$

$$\hat{x}(\alpha) = \alpha(x) = \alpha\left(\sum_i x^i e_i\right) = \sum_i x^i \alpha(e_i) = \sum_i x^i \hat{e}_i(\alpha)$$

~~$e_i(x)$~~

$$x = \sum x^i \hat{e}_i$$

$$(\hat{e}_i \otimes \hat{e}_j)(\alpha, \beta) = \alpha(e_i) \beta(e_j) \quad \langle \hat{e}_i | \langle e_i \rangle$$



$$(\hat{e}_i \otimes \hat{e}_j)(\alpha, \beta) = \hat{e}_i(\alpha) \hat{e}_j(\beta) = \alpha(e_i) \beta(e_j)$$

~~$e^i(x) = x^i$~~

$$\alpha(x) = \alpha\left(\sum x^i e_i\right)$$

$$= \sum_i x^i \alpha(e_i)$$

$$= \sum_i e^i(x) \alpha(e_i)$$

$$= \sum_i \alpha(e_i) e^i(x)$$

$$\alpha_i = \alpha(e_i)$$

$$\alpha = \sum_i \alpha(e_i) e^i$$

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(3)

$$X(e_i - e_j) = \alpha_i - \alpha_j$$

~~★~~  $(\hat{e}_i \otimes \hat{e}_j)(\alpha, \beta) = \alpha(e_i) \beta(e_j) = \alpha_i \beta_j$

 $(e^i \otimes e^j)(x, y) = x^i y^j$

Any bilinear

$S: (\mathbb{R}^n)^* \times (\mathbb{R}^n)^* \rightarrow \mathbb{R}$

$$\begin{aligned} S(\alpha, \beta) &= S\left(\sum_i \alpha_i e^i, \sum_j \beta_j e^j\right) \\ &= \sum_i \sum_j \alpha_i \beta_j S(e^i, e^j) \\ &= \sum_i \sum_j S^{ij} \hat{e}_i(\alpha) \hat{e}_j(\beta) \\ &= \sum_i \sum_j S^{ij} (\hat{e}_i \otimes \hat{e}_j)(\alpha, \beta) \end{aligned}$$

$S = \sum_i \sum_j S^{ij} (\hat{e}_i \otimes \hat{e}_j) \in T_2(\mathbb{R}^n)$

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$$S: \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$$

$$(g \cdot S) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

$$(g \cdot S)(\alpha, \beta) = S(g \cdot \alpha, g \cdot \beta)$$

$$(g \cdot \alpha)(x) = \alpha(g^T x)$$

$$\begin{aligned} (g \cdot \alpha)_i &= (g \cdot \alpha)(e_i) = \alpha(g^T e_i) \\ &= \alpha(\tilde{g}_i^T \otimes \tilde{e}_j) \\ &= \tilde{g}_i^T \alpha(e_j) \\ &= \tilde{g}_i^T \alpha_j \end{aligned}$$

Passive

$$f_i = \tilde{g}_i^T e_i$$

$$f_i = g e_i \quad (g \cdot S)(\alpha, \beta) = S(f \cdot \alpha)_i, f \cdot \beta)_j \quad \text{con } \alpha$$

$$\{e_i\}$$

$$\{f_i\}$$

$$\{\tilde{e}_i\}$$

$$\{\tilde{f}_i\}$$

$$\{e_i\}$$

$$\{f_i\}$$

$$= (g \cdot \alpha)_i (g \cdot \beta)_j S(e_i, e_j)$$

$$= S^{ij} \tilde{g}_i^T \alpha \tilde{g}_j^T \beta$$

$$\{S^{ij}\} \quad \tilde{S}^{kl} = S^{ij} \tilde{g}_i^T \tilde{g}_j^T$$

$$S^{ij} \tilde{g}_i^T \tilde{g}_j^T \hat{e}_k(\alpha) \hat{e}_l(\beta)$$

$$\uparrow \quad \text{Comp} \quad \text{Comp}$$

$$(g \cdot S)(\alpha, \beta) = S^{ij} \tilde{g}_i^T \tilde{g}_j^T (\hat{e}_k \otimes \hat{e}_l)(\alpha, \beta)$$

$$x \cdot e_i$$

$$x = \tilde{x}^i f_i$$

$$x \cdot e_i = x = \tilde{x}^i \tilde{g}_i^T \tilde{e}_j$$

$$x^i = \tilde{x}^i g^T$$

$$g \cdot S = S^{ij} \tilde{g}_i^T \tilde{g}_j^T (\hat{e}_k \otimes \hat{e}_l)$$

$$S = S^{ij} (\hat{e}_i \otimes \hat{e}_j)$$

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$$R: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$$

$$R = \sum R_{ij} (e^i \otimes e^j)$$

$$\begin{aligned} R_{ij} &= R(e_i, e_j) \stackrel{\downarrow}{=} -R(e_j, e_i) \\ &= -R_{ji} \end{aligned}$$

$$\begin{aligned} 2R &= \sum_{i,j} R_{ij} (e^i \otimes e^j) + \sum_{j \neq i} R_{ij} (e^j \otimes e^i) \\ &= \sum_{i,j} [R_{ij} (e^i \otimes e^j) - R_{ij} (e^j \otimes e^i)] \\ &= \sum_{i,j} R_{ij} [e^i \otimes e^j - e^j \otimes e^i] \end{aligned}$$

$$R = \sum_{i,j} R_{ij} (e^i \wedge e^j)$$

$$\mathbb{R}^n \cong \overline{T_p M}$$

$$e_i = \frac{\partial}{\partial x^i}$$

$$e^i = dx^i$$

$$T_p^2 \mathbb{R}^n$$

$$R_{abc}^{ij} (e^a \otimes e^b \otimes e^c \otimes e^i \otimes e^j)$$

$$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}^n)^* \times (\mathbb{R}^n)^*$$

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