Lecture 1: Matrix Notation & Algebra

The results we cover here are true in the context of a commutative ring with identity. We say $(R, +, \cdot)$ is a commutative ring if ① the addition $+ : R \times R \to R$ is commutative, associative, has an additive identity we call zero, and is closed under negation. ② the multiplication is associative, commutative and has a multiplicative identity 1. ③ addition and multiplication satisfy distributivity

$$a+b = b+a \qquad ab = ba$$
$$(a+b)+c = a+(b+c) \qquad (ab)c = a(bc)$$
$$0+a = a = a+0 \qquad 1a = a1 = a$$
$$a+(-a) = 0 \qquad a(b+c) = ab+ac$$

Defn/ If R is commutative ring with identity 1 then $x \in R$ is a <u>unit</u> if $\exists\, x^{-1} \in R$ s.t. $xx^{-1} = 1$. If each $x \in R$ with $x \neq 0$ is a unit then we say R is a <u>field</u>

Examples of fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ where p prime

Examples of commutative rings which are not fields:

$$\underbrace{C^0(\mathbb{R})}_{\text{continuous fncts.}}, \quad \underbrace{C^k(\mathbb{R})}_{k\text{-times differentiable}}, \quad \underbrace{\mathbb{Z}}_{\text{integers}}, \quad \underbrace{\mathbb{Z}_n}_{\text{integers modulo } n} \text{ where } n \text{ is composite}$$

Remark: I'll try to do some calculations in $\mathbb{Z}_n$ today in case you've never seen it.

Remark: we use R for a commutative ring with identity 1 for the remainder of this talk. Likewise $\mathbb{F}$ denotes a field (think $\mathbb{F} = \mathbb{R}$ if you wish to keep it real)

Def$^{\underline{n}}$/ An $m \times n$ matrix is an array of objects with m rows and n columns for some $m, n \in \mathbb{N}$. If A is an $m \times n$ matrix then A_{ij} denotes the object in the i^{th} row and j^{th} column.

$$A = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \qquad \begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array}$$

$$row_i(A) = [A_{i1}, \dots, A_{in}]$$

$$col_j(A) = \begin{bmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{bmatrix} \underset{def^{\underline{n}}}{=\!=} (A_{1j}, \dots, A_{mj})$$

hidden column notation, saves space.

If S is a set then $S^{m \times n}$ is the set of all $m \times n$ matrices of objects from S. If $m = n$ and $A \in S^{m \times n}$ then <u>A is square</u>.

$$A = \big[col_1(A) \,|\, \cdots \,|\, col_n(A) \big] = \begin{bmatrix} row_1(A) \\ \hline \vdots \\ \hline row_m(A) \end{bmatrix}$$

$$\boxed{\textbf{Def}^{\underline{n}}/ \text{ If } A, B \in S^{m \times n} \text{ then } A = B \text{ iff } A_{ij} = B_{ij} \text{ for all } 1 \le i \le m \text{ and } 1 \le j \le n.}$$

Def^n/ If $A, B \in \mathbb{R}^{m \times n}$ then $A+B,\ cA \in \mathbb{R}^{m\times n}$ for any $c \in \mathbb{R}$ where we define

$$(A+B)_{ij} = A_{ij} + B_{ij} \qquad \left(\begin{array}{c}A+B \text{ is the } \underline{\text{sum}} \\ \text{of } A \,\&\, B\end{array}\right)$$

$$(cA)_{ij} = cA_{ij} \qquad \left(\begin{array}{c}cA \text{ is the scalar} \\ \text{multiple of } A \text{ by } c\end{array}\right)$$

for all $1 \le i \le m$, $1 \le j \le n$.

In other words, we define matrix addition and scalar multiplication by componentwise rules.

Properties of Matrix Addition

Let $A, B, C \in \mathbb{R}^{m\times n}$ and $c_1, c_2 \in \mathbb{R}$ then

(1.) $(A+B)+C = A+(B+C)$

(2.) $A+B = B+A$

(3.) $c_1(A+B) = c_1A + c_1B$

(4.) $(c_1+c_2)A = c_1A + c_2A$

(5.) $(c_1c_2)A = c_1(c_2A)$

(6.) $1A = A$

Proof: let $1 \le i \le m$ and $1 \le j \le n$ and observe

(1.)
$$\begin{aligned}
((A+B)+C)_{ij} &= (A+B)_{ij} + C_{ij} && : \text{def}^n \text{ of } + \\
&= (A_{ij} + B_{ij}) + C_{ij} && : \text{def}^n \text{ of } + \\
&= A_{ij} + (B_{ij} + C_{ij}) && : \text{associativity of } + \text{ in } \mathbb{R} \\
&= A_{ij} + (B+C)_{ij} && : \text{def}^n \text{ of } + \\
&= (A+(B+C))_{ij} && : \text{def}^n \text{ of } +
\end{aligned}$$

Since this holds $\forall\, i,j$ it follows that $(A+B)+C = A+(B+C)$. //

Def$^{\underline{n}}$/ If $A \in S^{m\times n}$ then $A^T \in S^{n\times m}$ and $(A^T)_{ij} = A_{ji}$ for all $1 \leq j \leq m$, $1 \leq i \leq n$. We say A^T is the <u>transpose</u> of A.

Notice $(\text{row}_i(A))_j = A_{ij}$ whereas $(\text{col}_j(A))_i = A_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ provided $A \in \mathbb{R}^{m\times n}$

<u>Proposition</u>: if $A \in S^{m\times n}$ then

(i.) $(A^T)^T = A$

(ii.) $\text{row}_j(A^T) = (\text{col}_j(A))^T$

(iii.) $\text{col}_i(A^T) = (\text{row}_i(A))^T$

<u>Proof of (i.)</u> $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$ thus $(A^T)^T = A$.

<u>Proof of (ii.)</u> Suppose $1 \leq i \leq m$, $1 \leq j \leq n$ and consider

$$(\text{row}_j(A^T))_i = (A^T)_{ji} = A_{ij} = (\text{col}_j(A))_i = (\text{col}_j(A))^T_i$$

thus (ii.) follows since the above holds $\forall\, i = 1, 2, \dots, m$. //

<u>Proposition</u>: for $A, B \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}$

(i.) $(A+B)^T = A^T + B^T$

(ii.) $(cA)^T = cA^T$

MATRICES WITH A PARTICULAR SET OF SKILLS

Ok, to be honest, it's usually just one skill.

Def^n/ If $A \in \mathbb{F}^{n \times n}$ and $A^T = A$ then A is <u>SYMMETRIC</u> whereas if $A^T = -A$ then A is <u>ANTISYMMETRIC</u>.

$\boxed{E1}$ Suppose A is symetric and B is symmetric then $(A+B)^T = A^T + B^T = A + B$ hence $A+B$ is also symmetric.

Def^n/ If $A \in \mathbb{F}^{n \times n}$ then

(1.) A is <u>upper triangular</u> if $A_{ij} = 0$ for all $2 \leq i \leq n$ and $1 \leq j \leq n$ for which $i > j$.

<u>n=3</u> $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$, $A_{21} = 0$, $A_{31} = A_{32} = 0$

If A is upper-Δ with $A_{ii} = 0$ for $i = 1, 2, \dots, n$ then A is <u>strictly</u> upper-Δ

(2.) A is lower triangular if $A_{ij} = 0$ for all $1 \leq i \leq n$, $2 \leq j \leq n$ s.t. $i < j$.
Moreover A is strictly lower-Δ if A is lower-Δ with $A_{ii} = 0$ $\forall i = 1, 2, \dots, n$.

<u>Remark</u>: in the typed notes I have index proof about product of Δ-matrices being Δ.

Nilpotent Matrices

Defn/ If $A \in \mathbb{R}^{n \times n}$ has $A^{k-1} \neq 0$ and $A^k = 0$ then A is <u>nilpotent</u> order k.

Defn/ $N_m = [0 | e_1 | e_2 | \cdots | e_{m-1}] \in \mathbb{R}^{m \times m}$

Proposition: N_m is nilpotent order m

Proof: Let $m \in \mathbb{N}$. We'll use the matrix identity

$$A[v_1 | v_2 | \cdots | v_m] = [Av_1 | Av_2 | \cdots | Av_m]$$

as well as $Ae_k = \text{col}_k(A)$ repeatedly in what follows, we write $N = N_m$ for brevity

$$\begin{aligned} N_m N_m &= N[0 | e_1 | e_2 | \cdots | e_{m-1}] \\ &= [N0 | Ne_1 | Ne_2 | \cdots | Ne_{m-1}] \\ &= [0 | 0 | e_1 | \cdots | e_{m-2}] \end{aligned}$$

Then consider,

$$\begin{aligned} NNN &= N[0 | 0 | e_1 | \cdots | e_{m-2}] \\ &= [N0 | N0 | Ne_1 | Ne_2 | \cdots | Ne_{m-2}] \\ &= [0 | 0 | 0 | e_1 | \cdots | e_{m-3}] \end{aligned}$$

Continuing in this fashion,

$$N^k = [\underbrace{0 | \cdots | 0}_{k} | \underbrace{e_1 | \cdots | e_{m-k}}_{m-k}]$$

then $N^{m-1} = [0 | \cdots | 0 | e_1] \neq 0$ while $N^m = 0$. //

Remark: perhaps you can improve my argument and give an inductive proof.