# Lecture 2 : Matrices and Structure of Linear Systems

Let me begin with several proofs based on column-by-column argumentation. This is an alternative to the proof by indices technique (I use both)

**Claim:** If $I_n = [e_1|e_2|\cdots|e_n]$ is the $n \times n$ identity matrix and $X$ is an $m \times n$ matrix and $Y$ is an $n \times p$ matrix then

① $XI = X$

② $IY = Y$

Proof: ① 
$$\begin{aligned} XI &= X[e_1|e_2|\cdots|e_n] \\ &= [Xe_1|\cdots|Xe_n] &&: \text{column by column matrix mult. prop.} \\ &= [col_1(X)|\cdots|col_n(X)] &&: Ae_i = col_i(A) \\ &= X. \end{aligned}$$

② 
$$\begin{aligned} IY &= \begin{bmatrix} e_1^T \\ \hline \vdots \\ \hline e_n^T \end{bmatrix} Y &&: \text{notice } I = I^T \text{ thus } row_i(I) = (col_i(I))^T \\ &= \begin{bmatrix} e_1^T Y \\ \hline \vdots \\ e_n^T Y \end{bmatrix} &&: \text{row-by-row multiplication rule.} \\ &= \begin{bmatrix} row_1(Y) \\ \vdots \\ row_n(Y) \end{bmatrix} &&: e_i^T A = row_i(A). \\ &= Y. \end{aligned}$$

**Proposition**: Suppose $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ where $E_{ij} \in \mathbb{F}^{m\times n}$ for some field $\mathbb{F}$. Then suppose $\bar{E}_{ja} \in \mathbb{F}^{n\times p}$ is likewise the $n\times p$ matrix unit where $(\bar{E}_{ja})_{bc} = \delta_{jb}\delta_{ac}$. It follows that

$$E_{ij}\bar{E}_{ka} = \delta_{jk}\tilde{E}_{ia}$$

where $\tilde{E}_{ia}$ is the $m\times p$ matrix unit with 1 in the $(i,a)$-th entry and zero otherwise. Furthermore, matrix units can be constructed by the products of standard basis vectors and their transposes.

**Proof**: consider $E_{ij} \in \mathbb{F}^{m\times n}$ and denote $e_i \in \mathbb{F}^m = \mathbb{F}^{m\times 1}$ and $\bar{e}_j \in \mathbb{F}^n = \mathbb{F}^{n\times 1}$ then $e_i\bar{e}_j^T \in \mathbb{F}^{m\times n}$ and observe

$$\begin{aligned}(e_i\bar{e}_j^T)_{kl} &= e_k^T e_i \bar{e}_j^T \bar{e}_l \quad : \text{ select comp. by row - column mult.}\\ &= \delta_{ki}\delta_{jl}\\ &= (E_{ij})_{kl} \quad \therefore \boxed{E_{ij} = e_i\bar{e}_j^T}\end{aligned}$$

(since $k, l$ arbitrary we've established )

## Proof continued

We wish to study
$$E_{ij} \in \mathbb{F}^{m \times n}$$
$$\bar{E}_{ja} \in \mathbb{F}^{n \times p}$$
$$\tilde{E}_{ia} \in \mathbb{F}^{m \times p}$$

we can construct these matrix units with standard basis vectors denoted
$$e_i \in \mathbb{F}^m$$
$$\bar{e}_j \in \mathbb{F}^n$$
$$\tilde{e}_a \in \mathbb{F}^p$$
where, in view of last page's argument,
$$E_{ij} = e_i \bar{e}_j^T \text{ and } \bar{E}_{ja} = \bar{e}_j \tilde{e}_a^T \text{ and } \tilde{E}_{ia} = e_i \tilde{e}_a^T$$

Thus consider, for $1 \leq b \leq m$, $1 \leq c \leq p$,

$$E_{ij} \bar{E}_{ka} = e_i \underbrace{\bar{e}_j^T \bar{e}_k}_{\delta_{jk}} \tilde{e}_a^T = \delta_{jk} e_i \tilde{e}_a^T = \delta_{jk} \tilde{E}_{ia}$$

Thus $E_{ij} \bar{E}_{ka} = \delta_{jk} \tilde{E}_{ia}$ as claimed. //

## Socks - Shoes for Transpose

$$\boxed{\text{Given } A \in \mathbb{F}^{m\times n} \text{ and } B \in \mathbb{F}^{n\times p}, \quad (AB)^T = B^T A^T}$$

<u>Proof</u>: suppose $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$ then $AB \in \mathbb{F}^{m\times p}$ whereas $(AB)^T \in \mathbb{F}^{p\times m}$. Consider $1 \leq i \leq p$ and $1 \leq j \leq m$,

$$
\begin{aligned}
((AB)^T)_{ij} &= (AB)_{ji} && : \text{def}^{\underline{n}} \text{ of transpose} \\
&= \sum_{k=1}^{n} A_{jk} B_{ki} && : \text{def}^{\underline{n}} \text{ of mat. mult.} \\
&= \sum_{k=1}^{n} B_{ki} A_{jk} && : \text{scalar mult. commute in } \mathbb{F} \\
&= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} && : \text{def}^{\underline{n}} \text{ of transpose.} \\
&= (B^T A^T)_{ij} && : \text{def}^{\underline{n}} \text{ of matrix mult.}
\end{aligned}
$$

Therefore, $(AB)^T = B^T A^T$ since the calculation above shows $(AB)^T$ and $B^T A^T$ share all the same component entries. //

<u>Remark</u>: we can use PMI (proof by mathematical induction) to show $(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$ whenever $A_1, \ldots, A_n$ are multipliable matrices.

# SYSTEMS OF LINEAR EQUATIONS OVER $\mathbb{F}$ and the CCP



Given $m$-eq$^{n}$'s in $n$-unknowns over $\mathbb{F}$ we can formulate these as the matrix eq$^{n}$

$$Ax = b$$

coeff. matrix ($A$), solution vector ($x$), inhomogeneous term ($b$)
($b = 0$ makes it a homogeneous eq$^{n}$)

We learned the solution set is found by performing elementary row op. on $[A|b]$ and we saw there were elementary matrices $E_1, E_2, \dots, E_k$ for which

$$\text{rref}[A|b] = E_k \cdots E_2 E_1 [A|b] \quad (*)$$

We say $A$ and $B$ are row equivalent

$$A \xrightarrow[\text{row op's}]{\text{elem.}} B$$

$\boxed{\text{Th}^{m}\text{/ Given } M \text{ row equivalent to } N \text{ there exist elem. matrices } E_1, \dots, E_k \text{ s.t. } N = E_k \cdots E_2 E_1 M}$

(*) is the most used instance of row equiv.

## Thm/(CCP) (a.k.a the linear correspondence)

If $M$ and $N$ are row equivalent then any linear dependence of the columns of $M$ is also a linear dep. for the columns of $N$;

$$\sum_{i=1}^{n} a_i \, col_i(M) = 0 \iff \sum_{i=1}^{n} a_i \, col_i(N) = 0$$

for any $a_1, \dots, a_n \in \mathbb{F}$.

<u>Proof</u>: suppose $M$ row eq. to $N$. let $E = E_k \cdots E_2 E_1$ be the matrix for which $N = EM$ ($E_1, \dots, E_k$ elem. matrices).

Now assume $\sum_{i=1}^{n} a_i \, col_i(M) = 0$ for some $a_i \in \mathbb{F}$.

Multiply by $E$, and obtain

$$\sum_{i=1}^{n} a_i \, E \, col_i(M) = 0 \qquad (*)$$

However, $col_i(N) = E \, col_i(M)$ since

$$N = EM = E\,[col_1(M) | \cdots | col_n(M)] = [E col_1(M) | \cdots | E col_n(M)]$$

Thus, $(*)$ yields

$$\sum_{i=1}^{n} a_i \, col_i(N) = 0 \quad \therefore (\Rightarrow) \text{ true.}$$

By symmetry and invertibility of $E$ $(\Leftarrow)$ holds. //