LECTURE 3: VECTORSPACES & SUBSPACES

In this talk we'll define vector spaces and subspaces, give examples and discuss the subspace test Th$^{\underline{m}}$

Def$^{\underline{n}}$/ A vector space V over a field $\mathbb{F}$ is a set V together with a function $+ : V \times V \to V$ called <u>addition</u> and another function $\cdot : \mathbb{F} \times V \to V$ called <u>scalar</u> <u>multiplication</u>. These satisfy the following axioms:

$(V, +)$ an abelian group

(A1.) $x + y = y + x \;\; // \;\; \forall x, y \in V$.

(A2.) $(x+y)+z = x+(y+z) \;\; // \;\; \forall x, y, z \in V$

(A3.) $\exists\, 0 \in V$ s.t. $x + 0 = x = 0 + x \;\; // \;\; \forall x \in V$ (we say 0 is the zero vector)

(A4.) for each $x \in V$ there exists $-x \in V$ for which $x + (-x) = 0 = (-x) + x$.

(A5.) $1 \cdot x = x \;\; // \;\; \forall x \in V$ where $1 \in \mathbb{F}$

(A6.) $(ab) \cdot x = a \cdot (b \cdot x) \;\; // \;\; \forall x \in V,\ a, b \in \mathbb{F}$

(A7.) $a \cdot (x + y) = a \cdot x + a \cdot y \;\; // \;\; \forall x, y \in V,\ a \in \mathbb{F}$

(A8.) $(a+b) \cdot x = a \cdot x + b \cdot x \;\; // \;\; \forall x \in V,\ a, b \in \mathbb{F}$

redundant since $+$ and $\cdot$ are functions

(A9.) If $x, y \in V$ then $x + y$ is single element of V

(A10.) If $x \in V$ and $c \in \mathbb{F}$ then $c \cdot x$ is single element of V

We also write $V(\mathbb{F})$ to mean V is vect. space over $\mathbb{F}$.

Examples of Vector Spaces

The following sets form vector spaces using the standard addition of the objects and the standard scalar multiplication for the object.

$\boxed{E1}$ $V = \mathbb{R}^n$ with $\mathbb{F} = \mathbb{R}$

$\boxed{E2}$ $V = \mathbb{R}^n$ with $\mathbb{F} = \mathbb{Q}$; $V = \mathbb{R}^n(\mathbb{Q})$

$\boxed{E3}$ $V = \mathbb{R}^{m \times n}$ with $\mathbb{F} = \mathbb{R}$

$\boxed{E4}$ $V = \mathbb{R}^{m \times n}$ with $\mathbb{F} = \mathbb{Q}$; $V = \mathbb{R}^{m \times n}(\mathbb{Q})$

$\boxed{E5}$ $V = \mathbb{C}^{m \times n}$ with $\mathbb{F} = \mathbb{C}$

$\boxed{E6}$ $V = \mathbb{C}^{m \times n}$ with $\mathbb{F} = \mathbb{R}$; $V = \mathbb{C}^{m \times n}(\mathbb{R})$

$\boxed{E7}$ $V = \mathbb{C}^{m \times n}$ with $\mathbb{F} = \mathbb{Q}$; $V = \mathbb{C}^{m \times n}(\mathbb{Q})$

Remark: E1, E2 same set, different scalar mult. Likewise E3, E4 and E5, E6, E7.

Def^n/ Given field $\mathbb{F}$ we say $\mathbb{K} \subseteq \mathbb{F}$ is a <u>subfield</u> of $\mathbb{F}$ if $\mathbb{K}$ is a field w.r.t. to field arithmetic of $\mathbb{F}$.

Th^m/ Given $\mathbb{K} \subseteq \mathbb{F}$ and vector space $V(\mathbb{F})$ we find $V(\mathbb{K})$ is vector space formed with same $+$ and scalar mult. restricted to $\mathbb{K}$.

Addition of functions is well-known,

$\boxed{E8}$ Let S be a set then $f: S \to \mathbb{R}$ is a function and $V = \mathcal{F}(S, \mathbb{R})$ is set of all such functions. We define for $f, g \in \mathcal{F}(S, \mathbb{R})$ the functions $f+g, cf$ by

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in S.$$
$$(cf)(x) = cf(x) \quad \forall x \in S.$$

$\boxed{E9}$ Let W be vector space over $\mathbb{F}$ then $V = \mathcal{F}(S, W)$ is vector space via

$\boxed{E10}$ Polynomials

$$P_2(\mathbb{R}) = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$$

$$P_n(\mathbb{R}) = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\}$$

$$\mathbb{R}[x] = \{f(x) \mid f(x) \in P_n(\mathbb{R}) \text{ for some } n \in \mathbb{N} \cup \{0\}\}$$

$$f(x) \in \mathbb{R}[x] \Rightarrow f(x) = \sum_{n=0}^{\infty} c_n x^n$$

where finitely many $c_0, c_1, \ldots$ are nonzero.

We add by adding coefficients and scalar multiply by scalar multiplying coeff.

$\boxed{E11}$ $\underbrace{\mathbb{F}[x]}_{\text{poly. with coefficients in } \mathbb{F}}$ contains $P_n(\mathbb{F})$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

Def$^{\underline{n}}$/
$$f(x) + g(x) \equiv \sum_{n=0}^{\infty} (a_n + b_n) x^n$$
$$c f(x) \equiv \sum_{n=0}^{\infty} c a_n x^n$$

$\boxed{E12}$ Given vector spaces V and W over $\mathbb{F}$ then $V \times W = \{ (x,y) \mid x \in V, \ y \in W \}$ forms vector space over $\mathbb{F}$ via

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \ y_1 + y_2)$$
$$c \cdot (x, y) = (c \cdot x, \ c \cdot y)$$

$\boxed{E13}$ Given vector spaces $V_1, V_2, \dots, V_n$ over $\mathbb{F}$ can form $V = V_1 \times V_2 \times \dots \times V_n$ as vector space over $\mathbb{F}$ where $+$ and $\cdot$ are done componentwise like in $\boxed{E12}$

SUBSPACES

Def^{n}/ Let V be a vector space over field $\mathbb{F}$. If $W \subseteq V$ such that W is a vector space over $\mathbb{F}$ w.r.t. operations of V restricted to W then we say W is <u>subspace</u> of V and write $W \leq V$.

$\boxed{\text{E14}}$ $\{0\} \leq V$ and $V \leq V$ for any vector space V over $\mathbb{F}$. We say,

- $\{0\}$ is the <u>trivial</u> subspace
- $W \leq V$ with $W \neq V$ then W is a <u>proper</u> subspace

<u>**Th^{m}/ (Subspace Test)**</u>

Given $V(\mathbb{F})$ and $W \subseteq V$ with $W \neq \emptyset$. Then $W \leq V$ provided the following hold:

(1.) $x, y \in W \Rightarrow x + y \in W$;

(2.) $c \in \mathbb{F},\ x \in W \Rightarrow cx \in W$.

$\boxed{\text{E15}}$ $C^0(\mathbb{R}) \leq \mathcal{F}(\mathbb{R}, \mathbb{R})$ etc.

$\boxed{\text{E16}}$ $\text{Null}(A) = \{x \in \mathbb{F}^n \mid Ax = 0\}$ for $A \in \mathbb{F}^{m \times n}$ then we can use subspace test Th^{m} to prove $\text{Null}(A) \leq \mathbb{F}^n$ (we'll do in class)