# GEOMETRY & METRICS

I'll focus our attention on the case $\mathbb{F} = \mathbb{R}$ in this short note.

**Def'n** Given vector space $V$ over $\mathbb{R}$ we say $g : V \times V \to \mathbb{R}$ is a <u>metric</u> if

(i) $g$ is bilinear

(ii) $g$ is symmetric; $g(x,y) = g(y,x)$ $\forall x, y \in V$.

(iii.) $g$ is nondegenerate; $g(x,y) = 0$ for all $x \in V \Rightarrow y = 0$.

Then $(V, g)$ is a <u>geometry</u>.

This generalizes the concept of inner product over a real vector space since $\langle , \rangle : V \times V \to \mathbb{R}$ is both bilinear and symmetric and we can prove $\langle x, x \rangle \geq 0$ for $x \neq 0$ implies nondegeneracy.

Recall $\langle x, x \rangle = 0$ iff $x = 0$ for an inner product.

Thus $\langle x, y \rangle = 0$ $\forall x \in V \Rightarrow \langle y, y \rangle = 0 \Rightarrow y = 0$.

E1 Not every metric is an inner product.

Let $g(x,y) = -x_1 y_1 + x_2 y_2$ for all $x, y \in \mathbb{R}^2$

then $g(x,y) = x^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} y$ thus it is clear that $g$ is bilinear, note $g(x,y) = -x_1 y_1 + x_2 y_2 = -y_1 x_1 + y_2 x_2 = g(y,x)$

$$g(cx+y, z) = (cx+y)^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z$$

$$= c\, x^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z + y^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z$$

$$= c\, g(x,z) + g(y,z)$$

hence $g(z, cx+y) = g(cx+y, z) = c\, g(x,z) + g(y,z) = c\, g(z,x) + g(z,y)$.

E1 continued

Suppose $g(x,y) = x^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} y = 0$ for all $x \in \mathbb{R}^2$.

then $e_1^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} y = [-1, 0]\, y = -y_1 = 0 \quad \therefore \underline{y_1 = 0}$.

and $e_2^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} y = [0, 1]\, y = \underline{y_2 = 0}$. Hence $y = 0$

and we've shown $g$ is nondegenerate.

To see this is not an inner product notice

$$g(x,x) = -x_1^2 + x_2^2 \leq 0$$

whenever $x_1 \geq x_2$. Of particular interest are null-vectors (nonzero $x$ s.t. $g(x,x) = 0$)

these come from $x_1^2 = x_2^2 \Rightarrow \underline{x = c(1,1) \text{ or } c(1,-1)}$.

Pragmatically, it's nice to analyze a metric in terms of its matrix.

Def$^{n}$/ Given a metric $g : V \times V \to V$, if $\beta = \{v_1, v_2, \ldots, v_n\}$ is basis for $V$ then $g_{ij} = g(v_i, v_j)$ defines the matrix $G = [g_{ij}]$ whereas we use $G^{-1} = [g^{ij}]$ such that $\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_{ik}$.

Remark: when $V = \mathbb{R}^n$ then we can use the standard basis and $[g] = G$. For instance with E1 we have $[g] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. On the other hand, for Euclidean Space $g(x,y) = x \cdot y$ for $x, y \in \mathbb{R}^n$ we simply have $[g] = I_n$.

**Th$^m$/** If $G$ is the matrix of a metric $g$ w.r.t. basis $\beta$ then $G^T = G$ and $\det(G) \neq 0$

**Proof:** Let $G = [g(v_i, v_j)]$ be the matrix of the metric $g$ w.r.t. basis $\beta = \{v_1, \dots, v_n\}$.

Suppose $G[y]_\beta = 0$ for some $[y]_\beta \in \mathbb{R}^n$

then $y = \sum_{j=1}^{n} y^j v_j$ where $[y]_\beta = (y^1, y^2, \dots, y^n)$.

Notice $G[y]_\beta = 0$

$\Rightarrow e_i^T G [y]_\beta = 0 \quad \forall i = 1, 2, \dots, n$

$\Rightarrow \sum_{i=1}^{n} x^i e_i^T G [y]_\beta = 0 \quad \forall x^i \in \mathbb{R}$

$\Rightarrow [x]_\beta^T G [y]_\beta = 0 \quad \forall x \in V$

$\Rightarrow g(x, y) = 0 \quad \forall x \in V$ (by lemma below.)

$\Rightarrow y = 0$ by nondegeneracy of $g$

$\Rightarrow [y]_\beta = 0$

$\Rightarrow \text{Null}(G) = \{0\}$

$\Rightarrow \det(G) \neq 0$ and $G^{-1}$ exists. //

**Lemma:** given $x, y \in V$ and basis $\beta = \{v_1, \dots, v_n\}$ for which $[x]_\beta = (x^1, \dots, x^n)$ and $[y]_\beta = (y^1, \dots, y^n)$ if $G_{ij} = g(v_i, v_j)$ then $g(x, y) = [x]_\beta^T G [y]_\beta$.

$$[x]_\beta^T G [y]_\beta = \sum_{i=1}^{n} \sum_{j=1}^{n} x^i g(v_i, v_j) y^j = g\left(\sum_i x^i v_i, \sum_j y^j v_j\right)$$

$$= g(x, y). \; //$$

# Musical Morphisms

We have shown every metric on a finite dim'l real vector space $V$ has a matrix $G$ which is <u>symmetric</u> and invertible. This leads us to define isomorphisms between $V$ and $V^*$. Our notations of up/down indices inspire the terminology <u>$\#$ to raise index</u> whereas <u>$\flat$ to lower index</u> (thinking of components)

**Def$^n$/** If $\alpha \in V^*$ then $\#\alpha \in V$ is defined implicitly by $\alpha(x) = g(x, \#\alpha) \quad \forall x \in V$.

Likewise, $\flat : V \to V^*$ is defined by

$$(\flat(x))(y) = g(x,y)$$

for all $x, y \in V$.

We can prove $\#$ and $\flat$ are isomorphisms, indeed $\flat^{-1} = \#$. It's nice to find explicit formula for $\#$ and $\flat$ in the context of $\beta = \{v_1, \ldots, v_n\}$

<u>Given:</u> $x = \sum_i x^i v_i$ and $\alpha = \sum_i \alpha_i v^i$

Then we seek to find components of $\flat(x)$ and $\#(\alpha)$

$$\flat(x) = \sum_j (\flat(x))(v_j)\, v^j$$

$$= \sum_j g(x, v_j)\, v^j \quad \therefore (\flat(x))_j = g(x, v_j)$$

Continuing,

$$b(x) = \sum_j \underbrace{g(x, V_j)}\, v^j$$

$$b(x)_j = X_j \quad \text{(def}^n \text{ of index lowering of } x\text{)}$$

$$g(x, V_j) = g\Big(\sum_i x^i V_i, V_j\Big)$$

$$= \sum_i x^i\, g(V_i, V_j)$$

$$= \sum_{i=1}^n x^i g_{ij}$$

$$\boxed{\text{Def}^n \; X_j = \sum_{i=1}^n x^i g_{ij}}$$

swapping a contravariant index for a covariant index.

Likewise,

$$\#(\alpha) = \sum_{j=1}^n (\#(\alpha))^j V_j$$

defined implicitly by $\alpha(x) = g(x, \#\alpha)$ *

but we also know $\alpha = \sum \alpha(V_i)\, v^i$, so $\alpha_i = \alpha(V_i)$

we might as well set $x = V_i$ for insight in *

$$\alpha(V_i) = \alpha_i = g\Big(V_i, \sum_j (\#\alpha)^j V_j\Big)$$

$$\alpha_i = \sum_{j=1}^n (\#\alpha)^j g_{ij}$$

$$\Rightarrow \sum_{i=1}^n \alpha_i g^{ik} = \sum_{i,j=1}^n (\#\alpha)^j g_{ij} g^{ik} = \sum_{j=1}^n (\#\alpha)^j \delta_{jk}$$

$$(\#\alpha)^k = \sum_{i=1}^n \alpha_i g^{ik}$$

$$\boxed{\text{Def}^n \; \alpha^k = \sum_{i=1}^n \alpha_i g^{ik}}$$

raising the index

## Th²/ $b = \#^{-1}$ and $b: V \to V^*$ is linear trans.

**Proof:** $b: V \to V^*$ we defined by $b(x)(y) = g(x,y)$ $\forall x, y \in V$. We should prove $b(x) \in V^*$ for any $x \in V$.

Notice $b(x)(cy_1+y_2) = g(x, cy_1+y_2)$

$= c\, g(x,y_1) + g(x,y_2)$

$= c\, b(x)(y_1) + b(x)(y_2)$ $\quad \forall c \in \mathbb{R}$, $\forall y_1, y_2 \in V$

thus $b(x) \in V^*$. Then we also have $b(cx_1 + x_2) = c\,b(x_1) + b(x_2)$ since

$b(cx_1+x_2)(y) = g(cx_1+x_2, y)$

$= c\, g(x_1,y) + g(x_2,y)$

$= (c\, b(x_1) + b(x_2))(y)$ $\quad \forall y \in V$

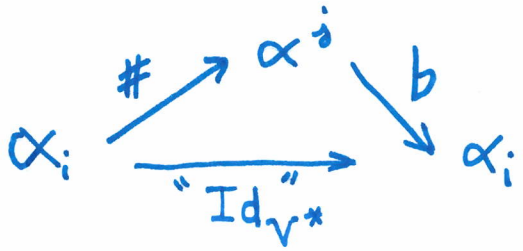
$\therefore$ $\underline{b(cx_1 + x_2) = c\,b(x_1) + b(x_2).}$

I've shown $b \in \mathcal{L}(V, V^*)$. It remains to prove $b = \#^{-1}$. I'll leave that for homework. I see two ways to show it,

(1.) w/o coordinates

(2.) gory index proof which will amount to ↴

$$\alpha_i = \sum_{j=1}^{n} \alpha^j g_{ij} = \sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_k g^{kj} g_{ij} = \sum_{k=1}^{n} \alpha_k \delta_{ki} = \alpha_i$$



# TENSOR MUSIC

Given a real geometry $(V, g)$ we can use the musical morphisms $\flat$ and $\#$ to convert multilinear maps from one form to another provided # of inputs matches.

① $T : V \times V \longrightarrow \mathbb{R}$ $\qquad R : V \times V^* \longrightarrow \mathbb{R}$

$S : V^* \times V^* \longrightarrow \mathbb{R}$

$$S(\alpha, \beta) = T(\#\alpha, \#\beta)$$

$$T(x, y) = S(\flat x, \flat y)$$

$$R(x, \alpha) = T(x, \#\alpha)$$

$$R(x, \alpha) = S(\flat x, \alpha)$$

If you study components of such tensors you'll find the up/down index juggling works in concert with the above constructions. For example,

$$S^{ij} = \sum_{k,l} T_{kl}\, g^{ki} g^{lj}$$

What happens to $\beta$ under $\flat$ or $\beta^*$ under $\#$?

$$\beta = \{ V_1, V_2, \ldots, V_n \}$$

$$\beta^* = \{ V^1, V^2, \ldots, V^n \} \quad \text{where} \quad V^i(V_j) = \delta_{ij}$$

① $V_i = \sum_{j=1}^{n} \delta_{ij} V_j$ if $x = V_i$ then $x^j = \delta_{ij}$

$$\flat(V_i) = \sum_{k,j=1}^{n} \delta_{ij} g_{jk} V^k = \sum_{k=1}^{n} g_{ik} V^k$$

② $V^i = \sum_{j=1}^{n} \delta_{ij} V^j$ if $\alpha = v^i$ then $\alpha_j = \delta_{ij}$

$$\begin{aligned} \#(v^i) &= \sum_{k=1}^{n} (\#(v^i))^k V_k \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} \delta_{ij} g^{jk} V_k \\ &= \sum_{k=1}^{n} g^{ik} V_k \end{aligned}$$

Th$^{m}$/ $\flat(V_i) = \sum_{k=1}^{n} g_{ik} V^k$

$\#(V^i) = \sum_{k=1}^{n} g^{ik} V_k$

I wonder what happens if we try to extend $\#$ and $\flat$ to $\otimes$ products...

# Sylvester's Theorem, or Law of Inertia?

Notice we have the matrix $G$ of a metric is both <u>symmetric</u> and <u>nondegenerate</u> consequently

$\exists$ eigenbasis $\beta = \{e_1, e_2, .., e_r, \bar{e}_1, \bar{e}_2, .., \bar{e}_s\}$

where we arranged eigenvalues $\lambda_1, \lambda_2, .., \lambda_r < 0$

and $\mu_1, \mu_2, .., \mu_s > 0$. Note $\lambda = 0$ is not an eigenvalue of $G$ as $G$ is invertible.

$$G e_i = \lambda_i e_i \qquad \forall i = 1, 2, .., r$$

$$G \bar{e}_j = \mu_j \bar{e}_j \qquad \forall j = 1, 2, .., s$$

Notice $\lambda_i = -|\lambda_i| = -\left(\sqrt{-\lambda_i}\right)^2$ for $i = 1, 2, .., r$

Likewise, $\mu_i = \left(\sqrt{\mu_i}\right)^2$ for $j = 1, 2, .., s$ thus

$$\frac{1}{\sqrt{-\lambda_i}} e_i^T G \frac{e_j}{\sqrt{-\lambda_j}} = \frac{1}{\sqrt{-\lambda_i}\sqrt{-\lambda_j}} \delta_{ij} \lambda_j = \frac{\lambda_i}{-\lambda_j} = -1$$

$$\frac{1}{\sqrt{\mu_i}} \bar{e}_i^T G \frac{\bar{e}_j}{\sqrt{\mu_j}} = \frac{1}{\sqrt{\mu_i}\sqrt{\mu_j}} \delta_{ij} \mu_j = \frac{\mu_i}{\mu_j} = 1$$

Consequently, if we construct basis

$$\gamma = \left\{ \frac{e_1}{\sqrt{-\lambda_1}}, \cdots, \frac{e_r}{\sqrt{-\lambda_r}}, \frac{\bar{e}_1}{\sqrt{\mu_1}}, \cdots, \frac{\bar{e}_s}{\sqrt{\mu_s}} \right\}$$

Then, the matrix of $g$ with respect to $\gamma$ is

$$G = \text{Diag}\left(\underbrace{-1, -1, .., -1}_{r}, \underbrace{1, 1, .., 1}_{s}\right)$$