LECTURE 4 : LINEAR COMBINATIONS & SPANS

This talk finishes a couple loose ends in §2.2 and covers §2.3

Def^{n}/ Let V, W be vector spaces over $\mathbb{F}$ a field. If a function $T: V \to W$ has

(1.) $T(x+y) = T(x) + T(y)$: additive

(2.) $T(cx) = cT(x)$: preserves scalar mult.

for $c \in \mathbb{F}$ and all $x, y \in V$ then T is a linear transformation. We say $T \in \mathcal{L}(V, W)$ and say $\mathcal{L}(V, V) = \mathcal{L}(V)$ is set of linear maps on V (set of <u>endomorphisms</u> of V)

$\boxed{E1}$ Can show $\mathcal{L}(V, W) \leq \mathcal{F}(V, W)$ by subspace test Th$^{\underline{m}}$. $0(x) = 0 \ \forall x \in V$ defines $0 \in \mathcal{L}(V, W) \neq \emptyset$. If $T, S \in \mathcal{L}(V, W)$ then can show $cT, T+S \in \mathcal{L}(V, W)$ for any $c \in \mathbb{F}$

$\boxed{E2}$ $V_p = \{ (p, v) \mid v \in \mathbb{F}^n \}$ defines vectors <u>based</u> at $p \in \mathbb{F}^n$ which we can denote by $(p, v) = v_p$ or $p + v$

Def^{n}/
$$(p, v) +_p (p, w) = (p, v+w) \quad \& \quad c \cdot_p (p, v) = (p, cv)$$
$$(p+v) +_p (p+w) = p + v + w \quad \& \quad c \cdot_p (p+v) = p + cv$$

Given $W \leq \mathbb{F}^n$ we can prove $W_p = \{ (p, w) \mid w \in W \} \leq V_p$.

[Application of E2, solution set of $Ax = b$ for $A \in \mathbb{F}^{m \times n}$ and $b \in \mathbb{F}^m$ has form $\{ x_p + x_h \mid x_h \in \text{Null}(A) \} = (\text{Null}(A))_{x_p}$ that is Sol^{n} Set $= x_p + \text{Null}(A) \leq V_{x_p}$ where $Ax_p = b$. Solution set is not subspace of $\mathbb{F}^n$ but it is of V_{x_p}.

Please study the other examples in §2.2, they should help reinforce how to apply the subspace test Th$^{\underline{m}}$ as well as some non-examples.

Lemma: Let $S \subseteq V$ a vector space over $\mathbb{F}$. Then a finite $\mathbb{F}$-linear combination of finite $\mathbb{F}$-linear combinations of vectors from S is once more a finite $\mathbb{F}$-linear combination of vectors from S

Proof: Suppose $S \subseteq V(\mathbb{F})$ where $\mathbb{F}$ is a field. Let $c_{ij} \in \mathbb{F}$ and $t_{ij} \in S$ $\quad s_i = \sum_{j=1}^{n_i} c_{ij} t_{ij}$ for $i = 1, 2, \dots, k$. Consider $b_1, b_2, \dots, b_k \in \mathbb{F}$ then

$$\underbrace{\sum_{i=1}^{k} b_i s_i}_{\text{linear combo of linear combos.}} = \sum_{i=1}^{k} b_i \left(\sum_{j=1}^{n_i} c_{ij} t_{ij} \right)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n_i} b_i c_{ij} t_{ij}$$

$$= \underbrace{b_1 c_{11} t_{11} + \cdots + b_k c_{kn_k} t_{kn_k}}_{\mathbb{F}\text{-linear combo of } S \text{ vectors.}}$$

Def$^{\underline{n}}$/ $V(\mathbb{F})$ a vector space, $S \subseteq V$. Then $\text{span}(S)$ is set of __all__ finite $\mathbb{F}$-linear combinations taken from S,

$$\text{span}(S) = \bigcup_{k \in \mathbb{N}} \left\{ \sum_{i=1}^{k} c_i v_i \;\middle|\; c_i \in \mathbb{F},\ v_i \in S \right\}$$

If $W = \text{span}(S)$ then S is __generating set__ of W. Also, $\text{span}(\emptyset) = \{0\}$.

When $|S| < \infty$ and $S = \{v_1, v_2, \dots, v_n\}$ then

$$\boxed{\text{Def}^{\underline{n}}/\ \operatorname{span}\{v_1, \dots, v_n\} = \left\{ \sum_{i=1}^{n} c_i v_i \ \middle|\ c_i \in \mathbb{F},\ i = 1, 2, \dots, n \right\}}$$

We write $\operatorname{span}_{\mathbb{F}}(S)$ to emphasize scalars taken from $\mathbb{F}$ if such emphasis is helpful.

Th$^{\underline{m}}$/ span(S) is subspace

Given $S \subseteq V$ where V is vector space over $\mathbb{F}$ then $\operatorname{span}(S) \leq V$. Furthermore, if $W \leq V$ and $S \subseteq W$ then $S \subseteq \operatorname{span}(S) \leq W$; that is $\operatorname{span}(S)$ is <u>smallest</u> subspace of V which contains S

Proof: apply Lemma to prove subspace test criterion, If $S = \emptyset$ then $\operatorname{span}\emptyset = \{0\} \leq V$. Else $S \neq \emptyset$ and if $x, y \in \operatorname{span}(S)$ and $c \in \mathbb{F}$ then $x+y,\ cx \in \operatorname{span}(S)$ and as $0 \cdot x = 0 \in \operatorname{span}(S) \neq \emptyset$ we find $\operatorname{span}(S) \leq V$.

Let $W \leq V$ and $S \subseteq W$. Then since W closed under $+$ and scalar mult. $\Rightarrow \operatorname{span}(S) \leq W$. To see S is subset of $\operatorname{span}(S)$ note $v \in S$ then $1 \cdot v = v \in \operatorname{span}(S)$, thus $S \subseteq \operatorname{span}(S)$. //

<u>Remark</u>: $x, y \in \operatorname{span}(S)$ then $x = \sum_{i=1}^{n} c_i s_i$ and $y = \sum_{j=1}^{m} b_j \tilde{s}_j$ where $s_i, \tilde{s}_j \in S$ then note

$$x + y = \sum_{i=1}^{n} c_i s_i + \sum_{j=1}^{m} b_j \tilde{s}_j \in \operatorname{span}(S)$$

$$cx = \sum_{i=1}^{n} (c c_i) s_i \in \operatorname{span}(S) \Rightarrow \text{can prove Th}^{\underline{m}} \text{ w/o Lemma.}$$

But, Lemma will be useful...

$\boxed{E3}$ $\mathbb{F}^n = \text{span}\{e_1, e_2, \dots, e_n\}$

$W = \text{span}\{e_1, \dots, e_k\} \leq \mathbb{F}^n$ for $1 \leq k \leq n$.

$\boxed{E4}$ $\mathbb{F}^{m\times n} = \text{span}\{E_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$

$W = \text{span}\{E_{11}, E_{12}, \dots, E_{1m}\} = \{[v|0|\cdots|0] \mid v \in \mathbb{F}^m\}$

$\boxed{E5}$ $P_2(\mathbb{R}) = \text{span}\{1, x, x^2\} \leq \mathbb{R}[x]$

$\boxed{E6}$ $W = \{(s+t, s-t, s, t) \mid s, t \in \mathbb{R}\}$

$= \text{span}\{(1,1,1,0), (1,-1,0,1)\} \leq \mathbb{R}^4$

$\boxed{E7}$ Solution set of $\underbrace{y''' = 0}$

$y'' = c_1$

$y' = c_1 t + c_2$

$y = \frac{1}{2}c_1 t^2 + c_2 t + c_3$

$\text{span}\{\frac{1}{2}t^2, t, 1\} = P_2(\mathbb{R})$.

Or, better, $\text{span}\{\frac{1}{2}t^2, t, 1\} \leq \mathcal{F}(\mathbb{R})$

$\boxed{E8}$ Given $A \in \mathbb{F}^{m\times n}$, there are 3 characteristic subspaces,

$\text{Col}(A) = \text{span}\{\text{col}_j(A) \mid j = 1, 2, \dots, n\} \leq \mathbb{F}^m$

$\text{Row}(A) = \text{span}\{\text{row}_i(A) \mid i = 1, 2, \dots, m\} \leq \mathbb{F}^{1\times n}$

$\text{Null}(A) \leq \mathbb{F}^n$ (by subspace test Th$^{\text{m}}$ last time)

CALCULATIONAL EXAMPLES

$\boxed{E9}$ $E_{11} \in \text{span}\{E_{12}+2E_{11}, E_{12}-E_{11}\}$? (YES)

We can work out $\frac{1}{3}(E_{12}+2E_{11}) - \frac{1}{3}(E_{12}-E_{11}) = E_{11}$

$\boxed{E10}$ work out generating set for 2x2 symmetric matrices, study $A^T = A$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to find

$$\text{span}\{E_{11}, E_{22}, E_{11}+E_{22}\} = \{A \in \mathbb{F}^{2\times 2} \mid A^T = A\}$$

Remark: time permitting, dive into LI and linear dependence.