Lecture 5 : Linear Independence & Bases

In this talk we study LI and spanning for abstract vector spaces over field $\mathbb{F}$. Ideally we'll cover a good amount of §2.4 and §2.5, my goal is to focus our Th^m's in finite dim'l context whereas the notes have some ∞-dim'l theory. (Ideally I'd expand §2.5 to include stand-alone proofs of theorems based on finite induction arguments)

We should cover,

- Def^n of LI, p. 62, Def^n 2.4.1
- Prop 2.4.3, if $0 \in S$ then S linearly dependent.
- Prop 2.4.4 | $v, w \neq 0$ — v, w linearly dependent $\Leftrightarrow$ $v = kw$ for some $k \in \mathbb{F}^\times$ $(k \neq 0)$
- Prop 2.4.5 | $$v_j \in \text{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\} \Leftrightarrow \{v_1, \dots, v_n\} \text{ not LI}$$
- Lemma 2.5.1 | S LI subset of vector space V and $v \in V$ with $v \notin S$ then $$S \cup \{v\} \text{ is LI} \Leftrightarrow v \notin \text{span}(S)$$
- Th^m (2.5.2) | Let S be LI subset of T where $\text{span}(T) = V$. Then $\exists \beta$ s.t. $S \subseteq \beta \subseteq T$ where β is both LI and $\text{span}(\beta) = V$.
- $\boxed{\text{Def}^n/\ V(\mathbb{F}) \text{ and } \beta \subseteq V \text{ s.t. } \beta \text{ LI and } \text{span}(\beta) = V \text{ then } \beta \text{ is \underline{basis} for } V.}$

The results given in Lemma 2.5.5, Cor 2.5.6 and Cor 2.5.7 ultimately lead to

Th$^{\underline{m}}$ (2.5.8) Any two bases of a vector space must have the same cardinality

Hence we define <u>dimension</u>

Def$^{\underline{n}}$/ Let V have basis β over $\mathbb{F}$. Then $\dim(V) = |\beta| = \#(\beta)$ is the <u>dimension</u> of V. If $\dim(V) < \infty$ then V is finite dimensional otherwise V is infinite dimensional

useful!

<u>Th$^{\underline{m}}$ 2.5.10</u>
LI $\Leftrightarrow$ span
given $\dim V = n$

Finally, and I'll probably cover this 1st in class,

Th$^{\underline{m}}$/ Given $S \subseteq V(\mathbb{F})$ TFAE,

(1.) S is LI

(2.) S allows equating coefficients;

$$\sum a_i s_i = \sum b_i s_i \Rightarrow a_i = b_i \ \forall i$$

given $s_i \in S$.

(3.) $\underbrace{\nexists\, v \in S \text{ s.t. } v \in \text{span}(S - \{v\})}$

in words, no vector in V can be written as linear combination of other vectors in S

thus $1 = 2a - b$ *and* $0 = a + b$. *Substitute* $a = -b$ *to find* $1 = 3a$ *hence* $a = \frac{1}{3}$ *and* $b = \frac{-1}{3}$. *Indeed,*

$$\frac{1}{3}(E_{12} + 2E_{11}) - \tfrac{1}{3}(E_{12} - E_{11}) = \tfrac{2}{3}E_{11} + \tfrac{1}{3}E_{11} = E_{11}.$$

Therefore, $E_{11} \in span\{E_{12} + 2E_{11}, E_{12} - E_{11}\}$.

Example 2.3.12. *Find a generating set for the set of symmetric* 2×2 *matrices. That is find a set* S *of matrices such that* $span(S) = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\} = W$. *There are many approaches, but I find it most natural to begin by studying the condition which defines* W. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ *and note*

$$A^T = A \ \Leftrightarrow\ \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ \Leftrightarrow\ A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

In summary, we find an equivalent way to express $A \in W$ *is as:*

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = aE_{11} + b(E_{12} + E_{21}) + dE_{22}.$$

Consequently $W = span\{E_{11},\ E_{12} + E_{21},\ E_{22}\}$ *and the set* $\{E_{11},\ E_{12} + E_{21},\ E_{22}\}$ *generates* W. *This is not unique, there are many other sets which also generate* W. *For example, if we took* $\bar{S} = \{E_{11},\ E_{12} + E_{21},\ E_{22}, E_{11} + E_{22}\}$ *then the span of* $\bar{S}$ *would still work out to* W.

2.4 linear independence

We have seen a variety of generating sets in the preceding section. In the last example I noted that if we added an additional vector $E_{11} + E_{22}$ then the same span would be created. The vector $E_{11} + E_{22}$ is **redundant** since we already had E_{11} and E_{22}. In particular, $E_{11} + E_{22}$ is a linear combination of E_{11} and E_{22} so adding it will not change the span. How can we decide if a vector is absolutely necessary for a span? In other words, if we want to span a subspace W then how do we find a **minimal spanning set**? We want a set of vectors which does not have any linear dependences. We say such vectors are linearly independent. Let me be precise[13]:

Definition 2.4.1.

Let $S \subseteq V$ a vector space over a field $\mathbb{F}$. The set of vectors S is Linearly Independent (LI) iff for any $\{v_1, v_2, \dots, v_k\} \subseteq S$ and $c_1, \dots, c_k \in \mathbb{F}$,

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \ \Rightarrow\ c_1 = c_2 = \cdots = c_k = 0.$$

Furthermore, to be clear, $\emptyset$ is LI. Moreover, if S is not LI then S is **linearly dependent**.

Example 2.4.2. *Let* $v = \cos^2(t)$ *and* $w = 1 + \cos(2t)$. *Clearly* v, w *are linearly dependent since* $w - 2v = 0$. *We should remember from trigonometry* $\cos^2(t) = \frac{1}{2}\big(1 + \cos(2t)\big)$.

Proposition 2.4.3.

[13] if you have a sense of deja vu here, it is because I uttered many of the same words in the context of $\mathbb{R}^n$. Notice, in contrast, I now consider the abstract case. We cannot use the CCP directly here

If S is a finite set of vectors which contains the zero vector then S is linearly dependent.

Proof: Let $\{\vec{0}, v_2, \dots v_k\} = S$ and observe that $1 \cdot \vec{0} = 0$ thus S is not linearly independent, that is, S is linearly dependent. $\square$

Proposition 2.4.4.

Let v and w be nonzero vectors.

$$v, w \text{ are linearly dependent } \Leftrightarrow \exists k \neq 0 \in \mathbb{F} \text{ such that } v = kw.$$

Proof: Let $v, w \neq 0$. Suppose $\{v, w\}$ is linearly dependent. Hence we find $c_1, c_2 \in \mathbb{F}$, not both zero, for which $c_1 v + c_2 w = 0$. Without loss of generality suppose $c_1 \neq 0$ then $v = -\frac{c_2}{c_1} w$ so identify $k = -c_2/c_1$ and $v = kw$. Conversely, if $v = kw$ with $k \neq 0$ then $v - kw = 0$ shows $\{v, w\}$ is linearly dependent. $\square$

It turns out the claim in the Proposition below does not generalize to the study of modules[14] so this is why we should use the Definition above as stated. Furthermore, this definition is oft used for calculations in our study of linear independence.

Proposition 2.4.5.

$v_j \in span\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}$ iff $\{v_1, v_2, \dots, v_k\}$ is a **linearly dependent** set.

Proof: ($\Rightarrow$) if $v_j \in span\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}$ then there exist $c_i \in \mathbb{F}$ for $i \neq j$ such that $v_j = \sum_{i \neq j} c_i v_i$ thus

$$c_1 v_1 + \cdots + c_{j-1} v_{j-1} - v_j + c_{j+1} v_{j+1} + \cdots + c_k v_k = 0$$

therefore $\{v_1, v_2, \dots, v_k\}$ is linearly dependent as there is a nontrivial linear combination whose sum is zero.

($\Leftarrow$) Suppose $\{v_1, v_2, \dots, v_k\}$ is linearly dependent. Then there exist $c_1, \dots, c_k \in \mathbb{F}$ for which

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

and at least one constant, say c_j, is nonzero. Then we can divide by c_j to obtain

$$\frac{c_1}{c_j} v_1 + \frac{c_2}{c_j} v_2 + \cdots + v_j + \cdots + \frac{c_k}{c_j} v_k = 0 \quad \Rightarrow \quad v_j = -\frac{c_1}{c_j} v_1 - \frac{c_2}{c_j} v_2 - \cdots - \widehat{v_j} - \cdots - \frac{c_k}{c_j} v_k$$

where $\widehat{v_j}$ denotes the deletion of v_j from the list. Thus $v_j \in span\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}$. $\square$

Given a set of vectors in $\mathbb{F}^n$ the question of LI is elegantly answered by the CCP. In examples that follow in this section we leave the comfort zone and study LI in abstract vector spaces. For now we only have brute force at our disposal. In other words, I'll argue directly from the definition without the aid of the CCP from the outset.

[14] A **module** is like a vector space, except, we use scalars from a ring. For example, if we consider pairs from $\mathbb{Z}/6\mathbb{Z}$ we have $2(3,0) + 3(0,2) = 0$ hence $\{(3,0), (0,2)\}$ is linearly dependent. But, note we cannot solve for $(3,0)$ as a multiple of $(0,2)$. In short, the solving for a vector criteria is special to using scalars from a field.

Example 2.4.6. *Suppose $f(x) = \cos(x)$ and $g(x) = \sin(x)$ and define $S = \{f, g\}$. Is S linearly independent with respect to the standard vector space structure on $\mathcal{F}(\mathbb{R})$? Let $c_1, c_2 \in \mathbb{R}$ and assume that*

$$c_1 f + c_2 g = 0.$$

It follows that $c_1 f(x) + c_2 g(x) = 0$ for each $x \in \mathbb{R}$. In particular,

$$c_1 \cos(x) + c_2 \sin(x) = 0$$

for each $x \in \mathbb{R}$. Let $x = 0$ and we get $c_1 \cos(0) + c_2 \sin(0) = 0$ thus $c_1 = 0$. Likewise, let $x = \pi/2$ to obtain $c_1 \cos(\pi/2) + c_2 \sin(\pi/2) = 0 + c_2 = 0$ hence $c_2 = 0$. We have shown that $c_1 f + c_2 g = 0$ implies $c_1 = c_2 = 0$ thus $S = \{f, g\}$ is a linearly independent set.

Example 2.4.7. *Let $f_n(t) = t^n$ for $n = 0, 1, 2, \dots$. Suppose $S = \{f_0, f_1, \dots, f_n\}$. Show S is a linearly independent subset of function space. Assume $c_0, c_1, \dots, c_n \in \mathbb{R}$ and*

$$c_0 f_0 + c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0. \qquad \star$$

I usually skip the expression above, but I'm including this extra step to emphasize the distinction between the function and its formula. The $\star$ equation is a function equation, it implies

$$c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n = 0 \qquad \star\star$$

for all $t \in \mathbb{R}$. Evaluate $\star\star$ at $t = 0$ to obtain $c_0 = 0$. Differentiate $\star^2$ and find

$$c_1 + 2c_2 t + \cdots + nc_n t^{n-1} = 0 \qquad \star^3$$

Evaluate $\star^3$ at $t = 0$ to obtain $c_1 = 0$. If we continue to differentiate and evaluate we will similarly obtain $c_2 = 0$, $c_3 = 0$ and so forth all the way up to $c_n = 0$. Therefore, $\star$ implies $c_0 = c_1 = \cdots = c_n = 0$.

Linear dependence in function space is sometimes a source of confusion for students. The idea of evaluation doesn't help in the same way as it just has in the two examples above.

Example 2.4.8. *Let $f(t) = t - 1$ and $g(t) = t + t^2$ is f linearly dependent on g? A common mistake is to say something like $f(1) = 1 - 1 = 0$ so $\{f, g\}$ is linearly independent since it contains zero. Why is this wrong? The reason is that we have confused the value of the function with the function itself. If $f(t) = 0$ for all $t \in \mathbb{R}$ then f is the zero function which is the zero vector in function space. Many functions will be zero at a point but that doesn't make them the zero function. To prove linear dependence we must show that there exists $k \in \mathbb{R}$ such that $f = kg$, but this really means that $f(t) = kg(t)$ for all $t \in \mathbb{R}$ in the current context. I leave it to the reader to prove that $\{f, g\}$ is in fact LI. You can evaluate at $t = 1$ and $t = 0$ to obtain equations for c_1, c_2 which have a unique solution of $c_1 = c_2 = 0$.*

Example 2.4.9. *Let $f(t) = t^2 - 1$, $g(t) = t^2 + 1$ and $h(t) = 4t^2$. Suppose*

$$c_1(t^2 - 1) + c_2(t^2 + 1) + c_3(4t^2) = 0 \qquad \star$$

A little algebra reveals,

$$(c_1 + c_2 + 4c_3)t^2 - (c_1 - c_2)1 = 0$$

Using linear independence of t^2 and 1 we find

$$c_1 + c_2 + 4c_3 = 0 \qquad \textit{and} \qquad c_1 - c_2 = 0$$

6

We find infinitely many solutions,

$$c_1 = c_2 \qquad \textit{and} \qquad c_3 = -\frac{1}{4}(c_1 + c_2) = -\frac{1}{2}c_2$$

Therefore, $\star$ allows nontrivial solutions. Take $c_2 = 1$,

$$1(t^2 - 1) + 1(t^2 + 1) - \frac{1}{2}(4t^2) = 0.$$

We can write one of these functions as a linear combination of the other two,

$$f = -g + \frac{1}{2}h.$$

Once we get past the formalities of the particular vector space structure it always comes back to solving systems of linear equations.

Remark 2.4.10.

We should keep in mind that in the abstract context statements such as "v and w go in the same direction" or "u is contained in the plane spanned by v and w" are not statments about ordinary three dimensional geometry. Moreover, you **cannot** write that $u, v, w \in \mathbb{R}^n$ unless you happen to be working with that rather special vector space. I caution the reader against the common mistake of trying to use column calculation techniques for things which are not columns. This is a very popular mistake in past years.

2.5 theory of dimension

Thanks to my brother William Cook of Appalachian State University, Boone North Carolina. This section of notes is nearly identical to a handout he gives to his Linear Algebra classes. In past editions of my notes, the focus was almost entirely on finite dimensional vector spaces. Here we learn that infinite dimensional vector spaces have dimension specified by cardinality. Let's go!

We can contrast the interplay of linear independence and spanning with subsets and supersets. If $S_1 \subseteq S_2$ and S_2 is linearly independent then S_1 is linearly independent. Likewise, if $S_1 \subseteq S_2 \subseteq W$ and $span(S_1) = W$ then $span(S_2) = W$; that is supersets of spanning sets still span[15]

Lemma 2.5.1.

Let S be a linearly independent subset of V, $v \in V$ such that $v \notin S$. Then $S \cup \{v\}$ is linearly independent if and only if $v \notin \text{span}(S)$.

Proof: Suppose $S \cup \{v\}$ is linearly independent. Then v cannot depend linearly on S. Thus $v \notin \text{span}(S)$.

Conversely, suppose $v \notin \text{span}(S)$. Suppose $S \cup \{v\}$ were linearly dependent. Then we could write $c_1v_1 + \cdots + c_\ell v_\ell + sv = 0$ for some $v_1, \dots, v_\ell \in S$ and scalars $c_1, \dots, c_\ell, s$ not all zero. Notice that $s \neq 0$ would imply $v = s^{-1}c_1v_1 + \cdots + s^{-1}c_\ell v_\ell \in \text{span}(S)$ (contradiction) so $s = 0$. But then $c_1v_1 + \cdots + c_\ell v_\ell = 0$ with not all $c_1, \dots, c_\ell$. This means S is linearly dependent! (contradiction). Thus $S \cup \{v\}$ must be independent. $\square$

[15]if we drop the condition $S_2 \subseteq W$ and merely suppose $S_1 \subseteq S_2$ then we can easily prove $span(S_1) \leq span(S_2)$

Theorem 2.5.2.

> Let S be a linearly independent subset of T where T spans V. Then there exists some β such that $S \subseteq \beta \subseteq T$ where β is both linearly independent and a spanning set for V. In other words, between any independent and spanning set, we can find a basis.

Proof: Consider the set $\mathcal{S} = \{S' \subseteq V \mid S \subseteq S' \subseteq T \text{ and } S' \text{ is linearly independent}\}$. Notice that S itself belongs to $\mathcal{S}$, so $\mathcal{S}$ is a non-empty set and is partially ordered by the subset relation. Suppose that we have a chain of elements in $\mathcal{S}$, say $\mathcal{C}$: this means that for all $S', S'' \in \mathcal{C}$ either $S' \subseteq S''$ or $S'' \subseteq S'$ (i.e., $\mathcal{C}$ is totally ordered by the subset relation). Consider $C = \cup\mathcal{C}$ (i.e., C is the union of all of the sets in our chain). Now every set in $\mathcal{C}$ contains S and is contained in T, so this is true of the union of such sets (i.e., $S \subseteq C \subseteq T$). Suppose $v_1, \dots, v_\ell \in C$ and $c_1, \dots, c_\ell \in \mathbb{F}$ such that $c_1v_1 + \cdots + c_\ell v_\ell = 0$. Then for each $i = 1, \dots, \ell$, we have $v_i \in S_i$ for some $S_i \in \mathcal{C}$. But $\mathcal{C}$ is a chain so we can (possibly after relabeling – without loss of generality) assume $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_\ell$. Thus $v_1, \dots, v_\ell \in S_\ell$. But ultimately S_ℓ is a set in $\mathcal{S}$ and so is linearly independent. Thus $c_1 = \cdots = c_\ell = 0$. Therefore, C is linearly independent. Therefore, $C \in \mathcal{S}$.

All of this shows that $\mathcal{S}$ is a non-empty set such that every chain is bounded above (by something in $\mathcal{S}$). Thus Zorn's Lemma applies and provides that $\mathcal{S}$ must contain a (possibly non-unique) maximal element. Call such an element β. We have then that β contains S, is contained in T, and is linearly independent. To finish our proof we just need to establish that β spans V.

Suppose T is not contained in span(β). This implies there is some $v \in T$ such that $v \notin \text{span}(\beta)$. So by our Lemma, $\beta' = \beta \cup \{v\}$ is linearly independent. But also $S \subseteq \beta \subseteq \beta' \subseteq T$ so $\beta' \in \mathcal{S}$. However, this is impossible because β was chosen to be maximal (i.e., it is not contained in something else in $\mathcal{S}$). Therefore, we must conclude that $T \subseteq \text{span}(\beta)$. Therefore, $V = \text{span}(T) \subseteq \text{span}(\beta) \subseteq V$ so that span$(\beta) = V$. Thus β is our desired linearly independent spanning set. □

We used Zorn's lemma in a non-trivial way. Zorn's result is equivalent to the Axiom of Choice. In fact, it can be shown that our theorem above is equivalent to the Axiom of Choice (we must use choice in some way to establish our theorem in general).

A set which both spans and is linearly independent is very special. We have a name for these:

Definition 2.5.3.

> If V is a vector space over the field $\mathbb{F}$ and $\beta \subseteq V$ such that β is linearly independent and $\text{span}_{\mathbb{F}}(\beta) = V$ then β is a **basis** for V.

It is difficult to stress just how important the following corollaries to above theorem are:

Corollary 2.5.4. *Basis can be created by extension of LI set or reduction of spanning set.*

> Every vector space has a basis. Every linearly independent set can be extended to a basis. Every spanning set can be shrunk down to a basis.

Proof: Applying the above theorem to the case $S = \emptyset$ and $T = V$ (these are always independent and spanning sets respectively), shows that we must have at least one basis. If S is linearly independent and we let $T = V$, we get that S is contained in some basis (i.e., every independent set can be extended to a basis). Finally, if T is a spanning set and we let $S = \emptyset$, we get a a basis β that is contained in T (i.e., spanning sets can be shrunk down to a basis). □

All of this leads us to the following linear algebra philosophy[16]: Bases are small enough spanning sets (small enough to be independent). Likewise, bases are big enough independent sets (big enough to span). We get the impression that independent sets must be no larger than spanning sets. This is true but requires some proof.

Lemma 2.5.5. *(The Exchange Lemma)*

Let $\alpha = \{v_1, \ldots, v_\ell\}$ be linearly independent and suppose T spans V. Then there exists some partition of $T = \{w_1, \ldots, w_\ell\} \coprod T'$ (i.e., T is a disjoint union of $\{w_1, \ldots, w_\ell\}$ and T') such that $T'' = \alpha \cup T'$ still spans V. In other words, if we select the right w_i's, we can swap out $\{w_1, \ldots, w_\ell\}$ with α and still span our vector space.

Proof: We proceed by induction. We can obviously swap zero vectors without difficulty, so our base case holds. Now suppose we have replaced $\{w_1, \ldots, w_k\}$ by $\{v_1, \ldots, v_k\}$ so that $T'' = \{v_1, \ldots, v_k\} \coprod T'$ spans V. Consider v_{k+1}. Since T'' spans V, $v_{k+1} = c_1v_1 + \cdots + c_kv_k + s_1u_1 + \cdots + s_mu_m$ for some $u_1, \ldots, u_m \in T'$ and $c_1, \ldots, c_k, s_1, \ldots, s_m \in \mathbb{F}$. Notice that if $s_1 = \cdots = s_m = 0$, then $v_{k+1} \in \text{span}\{v_1, \ldots, v_k\}$ implying that α is linearly dependent (contradiction). Therefore, at least one s_i is non-zero, say $s_m \neq 0$. For convenience call $s_m = s$ and $u_m = u$. Let $X = c_1v_1 + \cdots + c_kv_k + s_1u_1 + \cdots + s_{m-1}u_{m-1}$ so we have that $v_{k+1} = su + X$. Let $T' = \{u\} \coprod T_0$ (i.e., T_0 is all of T' except u). Notice that X is a linear combination of elements drawn from $\{v_1, \ldots, v_k\} \cup T_0$.
Therefore, in any linear combination, if it involves u, we can replace u with $s^{-1}v_{k+1} - s^{-1}X$. We now have that any linear combination of elements drawn from $\{v_1, \ldots, v_k\} \cup \{u\} \cup T_0$ can also be written as a linear combination of elements drawn from $\{v_1, \ldots, v_k\} \cup \{v_{k+1}\} \cup T_0$. In other words, we can swap out $w_{k+1} = u$ for v_{k+1} and our set will still span. The theorem now follows from induction. $\square$

Corollary 2.5.6.

If α is a finite linearly independent set and T is a spanning set, then $|\alpha| \leq |T|$. Moreover, if V is spanned by a finite set, it cannot have an infinite linearly independent subset.

Proof: By the Exchange Lemma we can replace elements of T with elements of α, so T must have at least as many elements as α. Now suppose that we have a finite spanning set T, say $|T| = m < \infty$. If we had an linearly independent set α with more than m elements, the Exchange Lemma would imply that we could replace $m+1$ elements of T with the first $m+1$ elements of α. This is absurd. $\square$

Corollary 2.5.7.

If any basis of a vector space is finite, all bases of that space are finite. Moreover, any two finite bases must have the same size.

Proof: Our previous corollary states that we cannot have a finite basis (a finite spanning set) and also an infinite basis (an infinite linearly independent subset).

Next, suppose α and β are finite bases for V. Then since α is finite linearly independent and β is a spanning set, our previous corollary says that $|\alpha| \leq |\beta|$. But also, β is finite linearly independent and α is a spanning set, so that $|\beta| \leq |\alpha|$. Therefore, $|\alpha| = |\beta|$. $\square$

[16] small as possible spanning set is also known as a minimal spanning set, likewise large as possible linearly independent set is also known as a maximally linearly independent set.

9

Theorem 2.5.8.

Any two bases of a vector space must have the same cardinality.

Proof: Let β and β' be bases for V. We already know that either both are infinite or both are finite. If both are finite, the corollary above establishes that $|\beta| = |\beta'|$. Therefore, we suppose that both are infinite and for sake of contradiction that $|\beta| < |\beta'|$.

Consider $v \in \beta$. Then since β' spans V, there exists $v_1, \dots, v_\ell \in \beta'$ and $c_1, \dots, c_\ell \in \mathbb{F}$ such that $v = c_1v_1 + \cdots + c_\ell v_\ell$. Define $F_v = \{v_1, \dots, v_\ell\}$ and so we have $v \in \text{span}(F_v)$. Let $\alpha = \bigcup_{v\in\beta} F_v$. Then α is a subset of β'. Moreover, for each $v \in \beta$ we have $v \in \text{span}(F_v) \subseteq \text{span}(\alpha)$ so $\beta \subseteq \text{span}(\alpha)$. Thus since β spans V, we have $V = \text{span}(\beta) = \text{span}(\alpha)$.

But this is problematic. Since α is a union of finite subsets indexed by β, we have that $|\alpha| \leq |\beta| \cdot \aleph_0 = |\beta|$ (since β is infinite). So $|\alpha| = |\beta| < |\beta'|$. Therefore α must be a proper subset of β'. Thus there is some $w \in \beta'$ that does not belong to α. Since β' is linearly independent, $\alpha \coprod \{w\}$ is independent. Therefore, $w \notin \text{span}(\alpha)$. But this means that $\text{span}(\alpha) \neq V$ (contradiction). Therefore, we must have that $|\beta| = |\beta'|$. □

Given all bases have the same size, we can make the following definition:

Definition 2.5.9.

Let V have a basis β. Then $\dim(V) = |\beta|$ is the *dimension* of V.

Finite dimensional spaces are special. For example, we have the following result.

Theorem 2.5.10.

Let V be a vector space of finite dimension n. Let $\beta = \{v_1, \dots, v_n\}$ be a subset of V of cardinality n. Then β is linearly independent if and only if β spans V.

Proof: If β is linearly independent, then it is contained in a basis, say γ. But $\beta \subseteq \gamma$ where $|\beta| = |\gamma| = n < \infty$ implies $\beta = \gamma$, so β is a basis. Likewise, if β spans V, then it contains a basis, say α. But $\alpha \subseteq \beta$ where $|\alpha| = |\beta| = n < \infty$ implies $\alpha = \beta$, so β is a basis. □

The above theorem does not work for infinite dimensional spaces. For example, $\{1, x^2, x^4, \dots\}$ is a countably infinite, linearly independent subset of the countably infinite dimensional space $\mathbb{R}[x]$, yet it fails to span. On the other hand, $\{2, 1, x, x^2, \dots\}$ is a countably infinite, spanning set for $\mathbb{R}[x]$, yet it fails to be independent. Having one property plus the "right size" isn't enough to force sets to be bases if we work in infinite dimensional spaces.