LECTURE 34: REAL ASSOCIATIVE ALGEBRAS

We discussed the definition and basic representation theory of unital-associative algebras of finite dimension. Essentially we covered §4 of my paper from 2017, which is currently unpublished, INTRODUCTION TO $\mathcal{A}$-CALCULUS. I'll review the main points here and include the relevant pages from the paper ↷

Defn/ $l_\alpha : \mathcal{A} \to \mathcal{A}$ is left multiplication by $\alpha \in \mathcal{A}$ given by $l_\alpha(v) = \alpha * v$ for all $v \in \mathcal{A}$

Defn/ $T : \mathcal{A} \to \mathcal{A}$ is RIGHT-$\mathcal{A}$-linear if $T(v * w) = T(v) * w$ for all $v, w \in \mathcal{A}$. We say $\mathcal{R}_\mathcal{A}$ is the collection of all such T is the REGULAR REPRESENTATION of $\mathcal{A}$

Defn/ Given basis $\beta = \{v_1, v_2, \dots, v_n\}$ for $\mathcal{A}$ we define the matrix regular representation of $\mathcal{A}$ with respect to β by

$$\mathcal{M}_\mathcal{A}(\beta) = \{ [T]_{\beta,\beta} \mid T \in \mathcal{R}_\mathcal{A} \}$$

When $\mathcal{A} = \mathbb{R}^n$ and $\beta = \{e_1, \dots, e_n\}$ is standard basis we simply write $\mathcal{M}_\mathcal{A} = \{ [T] \mid T \in \mathcal{R}_\mathcal{A} \}$

We discussed that $\mathcal{R}_\mathcal{A}$ is subalgebra of $gl(\mathcal{A})$ under composition and $\mathcal{M}_\mathcal{A}(\beta)$ is subalgebra of $\mathbb{R}^{n \times n}$ w.r.t. matrix multiplication (here $\dim(\mathcal{A}) = n$ is assumed)

Proof continued

$$Ker(\Psi) = \{ \alpha \in A \mid \Psi(\alpha) = Id_A \}$$

Thus $\alpha \in Ker(\Psi) \Rightarrow l_\alpha(x) = Id(x) \quad \forall x \in A$
and setting $x = 1$ yields $\alpha * 1 = 1 \Rightarrow \alpha = 1$.
Thus $Ker(\Psi) = \{1\} \Rightarrow \Psi : A \to \mathcal{R}_A$ is a linear bijection which preserves multiplication, moreover $\Psi(1) = l_1 = Id = 1_{\mathcal{R}_A}$ hence Ψ serves as an algebra isomorphism of $A \approx \mathcal{R}_A$.

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To show $\mathcal{R}_A \approx M_A(\beta)$ we can show $\Phi : A \to M_A(\beta)$ defined by $\Phi(T) = [T]_{\beta,\beta}$ is a linear bijection for which $\Phi(T_1 \circ T_2) = [T_1 \circ T_2]_{\beta,\beta} = [T_1]_{\beta\beta}[T_2]_{\beta\beta} = \Phi(T_1)\Phi(T_2)$
and $\Phi(Id) = [Id]_{\beta,\beta} = I_n \in \mathbb{R}^{n\times n}$ $\therefore$ Φ isomorphism and $\mathcal{R}_A \approx M_A(\beta)$.

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Def^n/ The mapping $\mathcal{M} : A \longrightarrow M_A(\beta)$ given by

$$\mathcal{M}(\alpha) = [l_\alpha]_{\beta,\beta} = \big[[\alpha * v_1]_\beta \big| [\alpha * v_2]_\beta \big| \cdots \big| [\alpha * v_n]_\beta \big]$$

is the <u>matrix representation of α w.r.t. β</u>.

Notice, $\mathcal{M}$ is composition of linear bijections and is thus a linear bijection and

$$\mathcal{M}(\alpha * \gamma) = [l_{\alpha*\gamma}]_{\beta,\beta} = [l_\alpha \circ l_\gamma]_{\beta,\beta} = [l_\alpha]_{\beta\beta}[l_\gamma]_{\beta,\beta}$$
$$= \mathcal{M}(\alpha)\mathcal{M}(\gamma).$$

We examined the isomorphism of $\mathcal{A}$, $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}(\beta)$ as real associative algebras of finite dimension. Let me recap that discussion here, but this time I'll follow the notation of my paper.

$\boxed{\text{Th}^{m}\text{(4.8) If } \beta \text{ is a basis for } \mathcal{A} \text{ then } \mathcal{A} \approx \mathcal{R}_{\mathcal{A}} \approx \mathcal{M}_{\mathcal{A}}(\beta)}$

Proof: consider $\beta = \{v_1, v_2, \dots, v_n\}$ a basis for $\mathcal{A}$.

Let $\Psi : \mathcal{A} \longrightarrow \mathcal{R}_{\mathcal{A}}$ be defined by $\Psi(\alpha) = l_\alpha$.

If $\alpha, \beta \in \mathcal{A}$ and $c \in \mathbb{R}$ and $v \in \mathcal{A}$ then

$$\begin{aligned}
(\Psi(\alpha + c\beta))(v) &= l_{\alpha+c\beta}(v) \\
&= (\alpha + c\beta) * v \\
&= \alpha * v + c(\beta * v) \\
&= (\Psi(\alpha) + c\Psi(\beta))(v)
\end{aligned}$$

Hence $\Psi(\alpha + c\beta) = \Psi(\alpha) + c\Psi(\beta)$ thus Ψ is $\mathbb{R}$-linear and we proved $l_\alpha \in \mathcal{R}_{\mathcal{A}}$ hence $\Psi : \mathcal{A} \to \mathcal{R}_{\mathcal{A}}$ is into. Indeed Ψ is a surjection since $T \in \mathcal{R}_{\mathcal{A}}$ has $T = l_{T(1)}$ where $1 = 1_{\mathcal{A}} \in \mathcal{A}$ (the identity in $\mathcal{A}$). It remains to show Ψ preserves the multiplicative structure of $\mathcal{A}$,

$$\begin{aligned}
(\Psi(\alpha * \beta))(v) &= l_{\alpha*\beta}(v) \\
&= (\alpha * \beta) * v \\
&= \alpha * (\beta * v) \\
&= l_\alpha(l_\beta(v)) \\
&= (l_\alpha \circ l_\beta)(v) \\
&= (\Psi(\alpha) \circ \Psi(\beta))(v) \quad \forall v \in \mathcal{A}
\end{aligned}$$

Thus $\Psi(\alpha * \beta) = \Psi(\alpha) \circ \Psi(\beta)$. Finally we show Ψ injective

Proof continued:

Since isomorphism is an equivalence relation it would have sufficed to show $A \approx \mathcal{R}_A$ and $\mathcal{R}_A \approx \mathcal{M}_A(\beta)$ but I think it is instructive to try to find explicit isomorphisms and their inverse maps. Let's just focus on formulas w/o proof

$$\Psi : A \longrightarrow \mathcal{R}_A \;, \quad \Psi(\alpha) = l_\alpha$$

$$\Psi^{-1} : \mathcal{R}_A \longrightarrow A \;, \quad \underbrace{\Psi^{-1}(T) = T(1)}$$

$$\Psi(\Psi^{-1}(T)) = \Psi(T(1)) = l_{T(1)}$$

$$l_{T(1)}(x) = T(1) * x = T(1 * x) = T(x)$$

$$\therefore \; l_{T(1)} = T.$$

$$\mathcal{M} : A \longrightarrow \mathcal{M}_A(\beta), \quad \mathcal{M}(\alpha) = [l_\alpha]_{\beta,\beta}$$

$$\underbrace{\mathcal{M}^{-1} : \mathcal{M}_A(\beta) \longrightarrow A}_{\text{this requires some thinking}}$$

this requires some thinking

in general, however, if $v_1 = 1_A$ then $\mathcal{M}^{-1}(B) = \Phi_\beta^{-1}(col_1(B))$

or easier still, if $\beta = \{e_1, e_2, .., e_n\}$ where $A = \mathbb{R}^n$ and $e_1 = 1_A$ then $\mathcal{M}^{-1}(B) = col_1(B)$.

Suppose $[1_A]_\beta = \vec{I} \in \mathbb{R}^n$ in the sense that $\vec{I} = (c_1, c_2, \dots, c_n)$ where

$$1_A = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

then we can use $\vec{I}$ to formulate $M^{-1} : M_A(\beta) \to A$

$$M^{-1}(B) = \Phi_\beta^{-1}(\underbrace{B}\vec{I})$$

$$= \Phi_\beta^{-1}(c_1 col_1(B) + c_2 col_2(B) + \cdots + c_n col_n(B))$$

Consider,

$$M(M^{-1}(B)) = M(\Phi_\beta^{-1}(B\vec{I}))$$
$$= [\Phi_\beta^{-1}(B\vec{I})]_{\beta,\beta}$$

where $[y_1 w_1 + \cdots + y_m w_m]_\gamma = (y_1, \dots, y_m) \in \mathbb{R}^m$ is the usual γ-coordinate map. In many of the applications we study the vector spaces V and W are $\mathbb{R}^n$ and it is our custom to use $e_1, \dots, e_n$ to denote the **standard basis** where $(e_i)_j = \delta_{ij}$. In this special case, we note the Jacobian matrix simply by the standard matrix of the differential;

$$[d_pF] = [d_pF(e_1)|\cdots|d_pF(e_n)] = \left[\frac{\partial F}{\partial x^1}\bigg|\cdots\bigg|\frac{\partial F}{\partial x^n}\right]. \tag{7}$$

4 Real linear associative algebras

A vector space paired with a multiplication forms an *algebra*.

Definition 4.1. *Let $\mathcal{A}$ be a finite-dimensional real vector space paired with a function $\star : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is called* **multiplication**. *In particular, the multiplication map satisfies the properties below:*

(i.) bilinear: *$(cx + y) \star z = c(x \star z) + y \star z$ and $x \star (cy + z) = c(x \star y) + x \star z$ for all $x, y, z \in \mathcal{A}$ and $c \in \mathbb{R}$,*

(ii.) associative: *for which $x \star (y \star z) = (x \star y) \star z$ for all $x, y, z \in \mathcal{A}$ and,*

(iii.) unital: *there exists $\mathbb{1} \in \mathcal{A}$ for which $\mathbb{1} \star x = x$ and $x \star \mathbb{1} = x$.*

We say $x \in \mathcal{A}$ is an $\mathcal{A}$**-number**. *If $x \star y = y \star x$ for all $x, y \in \mathcal{A}$ then $\mathcal{A}$ is* **commutative**.

When there is no ambiguity we use $1 = \mathbb{1}$ and we replace $\star$ with juxtaposition; $xy = x \star y$. We assume $\mathcal{A}$ is an associative algebra of finite dimension over $\mathbb{R}$ throughout the remainder of this paper. In the commutative case there is no need to distinguish between *left* and *right* properties. However, we allow the possibility that $\mathcal{A}$ be **noncommutative** at this point in our development.

If $\alpha \in \mathcal{A}$ then $\ell_\alpha(x) = \alpha \star x$ is a **left-multiplication** map on $\mathcal{A}$. It is a **right-$\mathcal{A}$-linear** as:

$$\ell_\alpha(x \star y) = \alpha \star (x \star y) = (\alpha \star x) \star y = \ell_\alpha(x) \star y. \tag{8}$$

Likewise, $r_\alpha(x) = x \star \alpha$ is a **right-multiplication** map on $\mathcal{A}$. It is a **left-$\mathcal{A}$-linear** as:

$$r_\alpha(x \star y) = (x \star y) \star \alpha = x \star (y \star \alpha) = x \star r_\alpha(y). \tag{9}$$

Notice, associativity of $\star$ is given by $\ell_\alpha \circ r_\beta = r_\beta \circ \ell_\alpha$ for all $\alpha, \beta \in \mathcal{A}$. From Equation 8 we see that every left-multiplication map is right-$\mathcal{A}$-linear. In fact, if $\mathbb{1} \in \mathcal{A}$ then every right-$\mathcal{A}$-linear map on $\mathcal{A}$ is a left multiplication by a particular element of $\mathcal{A}$.

Theorem 4.2. *If $T : \mathcal{A} \to \mathcal{A}$ is a right-$\mathcal{A}$-linear map then there exists a unique $\alpha \in \mathcal{A}$ for which $T = \ell_\alpha$.*

Proof: let $T \in \mathcal{R}_\mathcal{A}$ and consider $T(x) = T(\mathbb{1} \star x) = T(\mathbb{1}) \star x$ for each $x \in \mathcal{A}$. Therefore, $T = \ell_{T(\mathbb{1})}$. $\square$

The simple calculation above is key to understanding generalized Cauchy Riemann equations.

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Definition 4.3. *Let $\mathcal{R}_\mathcal{A}$ define the set of all right-$\mathcal{A}$-linear transformations on $\mathcal{A}$. If $T \in \mathcal{R}_\mathcal{A}$ then $T : \mathcal{A} \to \mathcal{A}$ is a $\mathbb{R}$-linear transformation for which $T(x \star y) = T(x) \star y$ for all $x, y \in \mathcal{A}$*

Recall the sum, scalar multiple and composition of endomorphisms is once more an endomorphism. Moreover, addition, scalar multiplication and composition of transformations are known to be bilinear and associative. Furthermore, if $Id(x) = x$ for all $x \in V$ then observe $Id = \mathbb{1}$ for the operation of composition. In summary, $\mathrm{gl}(V) = \{T : V \to V \mid T \text{ linear transformation}\}$ forms the **general linear algebra on** V. In the context of $V = \mathcal{A}$, the subset $\mathcal{R}_\mathcal{A} \subseteq \mathrm{gl}(\mathcal{A})$ is special:

Theorem 4.4. *The set $\mathcal{R}_\mathcal{A}$ is a subalgebra of $gl(\mathcal{A})$; that is, $\mathcal{R}_\mathcal{A} \leq gl(\mathcal{A})$*[3].

Proof: notice that $Id(x \star y) = x \star y = Id(x) \star y$ hence $Id \in \mathcal{R}_\mathcal{A}$. To show $\mathcal{R}_\mathcal{A}$ is a subalgebra of $\mathrm{gl}(\mathcal{A})$ it remains to show the sum, scalar multiple and composite of right-$\mathcal{A}$-linear maps in once again in $\mathcal{R}_\mathcal{A}$. Let $T_1, T_2 \in \mathcal{R}_\mathcal{A}$ and $c \in \mathbb{R}$. Suppose $x, y \in \mathcal{A}$ and consider:

$$\begin{aligned} (cT_1 + T_2)(x \star y) &= c(T_1)(x \star y) + T_2(x \star y) \\ &= cT_1(x) \star y + T_2(x) \star y \\ &= (cT_1(x) + T_2(x)) \star y \\ &= (cT_1 + T_2)(x) \star y. \end{aligned} \tag{10}$$

Likewise, applying right-linearity of T_2 then T_1 yields:

$$(T_1 \circ T_2)(x \star y) = T_1(T_2(x \star y)) = T_1(T_2(x) \star y)) = T_1(T_2(x)) \star y = (T_1 \circ T_2)(x) \star y. \tag{11}$$

Hence $T_1 \circ T_2 \in \mathcal{R}_\mathcal{A}$. $\square$

The result above shows that for any algebra $\mathcal{A}$ we immediately obtain a related algebra of linear transformations. Given a choice of basis, we also can trade $\mathcal{A}$ for a particular set of matrices known as the **regular representation**.

Definition 4.5. *Let $\mathcal{A}$ have basis β then the* **regular representation** *with respect to β is*

$$M_\mathcal{A}(\beta) = \{[T]_{\beta,\beta} \mid T \in \mathcal{R}_\mathcal{A}\}.$$

In the case $\mathcal{A} = \mathbb{R}^n$ we may forego the β notation and write

$$M_\mathcal{A} = \{[T] \mid T \in \mathcal{R}_\mathcal{A}\}$$

for the **regular representation** *of $\mathcal{A}$.*

The regular representation of $\mathrm{gl}(V)$ is simply $\mathbb{R}^{n \times n}$ matrices given $\dim(V) = n$. This is immediate from the fact that any matrix may appear as the matrix of an endomorphism. Just as $\mathcal{R}_\mathcal{A} \leq \mathrm{gl}(\mathcal{A})$ we likewise find $\mathrm{M}_\mathcal{A}(\beta) \leq \mathbb{R}^{n \times n}$. In $\mathbb{R}^{n \times n}$ the algebra multiplication is simply matrix multiplication and $\mathbb{1}$ is the $n \times n$ identity matrix.

Theorem 4.6. *For any choice of β, the set $M_\mathcal{A}(\beta)$ is a subalgebra of $\mathbb{R}^{n \times n}$.*

[3] we intend the notation $\mathcal{A} \leq \mathcal{B}$ to indicate $\mathcal{A}$ is a **subalgebra** of $\mathcal{B}$.

Proof: since $[Id]_{\beta,\beta} = I$ it follows that $\mathbb{1} \in \mathrm{M}_{\mathcal{A}}(\beta)$. It remains to show that $\mathrm{M}_{\mathcal{A}}(\beta)$ is closed under addition, scalar multiplication and matrix multiplication. Assume $A, B \in \mathrm{M}_{\mathcal{A}}(\beta)$ and $c \in \mathbb{R}$. Observe, by definition, there exist $S, T \in \mathcal{R}_{\mathcal{A}}$ for which $A = [S]_{\beta,\beta}$ and $B = [T]_{\beta,\beta}$. Since the matrix of a linear combination of operators is the linear combination of the matrices of said operators we have:

$$cA + B = c[S]_{\beta,\beta} + [T]_{\beta,\beta} = [cS + T]_{\beta,\beta} \tag{12}$$

but, we know $S, T \in \mathcal{R}_{\mathcal{A}}$ hence by Theorem 4.4 $cS + T \in \mathcal{R}_{\mathcal{A}}$ which shows $[cS + T]_{\beta,\beta} \in \mathrm{M}_{\mathcal{A}}$ and thus $cA + B \in \mathrm{M}_{\mathcal{A}}$. Finally, recall matrix multiplication was defined precisely so the identity below holds true:

$$AB = [S]_{\beta,\beta}[T]_{\beta,\beta} = [S \circ T]_{\beta,\beta}. \tag{13}$$

Observe, Theorem 4.4 indicates that $S, T \in \mathcal{R}_{\mathcal{A}}$ implies $S \circ T \in \mathcal{R}_{\mathcal{A}}$. Therefore, $AB \in \mathrm{M}_{\mathcal{A}}$ and we conclude $\mathrm{M}_{\mathcal{A}}$ forms a subalgebra of $\mathbb{R}^{n \times n}$. $\square$

Different basis choices give different **matrix** regular representations. We use the change of basis theorem from linear algebra to relate the representations:

Theorem 4.7. *If β, γ are bases for $\mathcal{A}$ then $M_{\mathcal{A}}(\beta)$ and $M_{\mathcal{A}}(\gamma)$ are conjugate subalgebras.*

Proof: Let β, γ be basis for $\mathcal{A}$. If $T : \mathcal{A} \to \mathcal{A}$ is a linear transformation then we know from linear algebra that there exists an invertible change of basis matrix P for which $[T]_{\beta,\beta} = P^{-1}[T]_{\gamma,\gamma}P$. Thus, if $A \in \mathrm{M}_{\mathcal{A}}(\beta)$ then $A = [T]_{\beta,\beta} = P^{-1}[T]_{\gamma,\gamma}P \in P^{-1}\mathrm{M}_{\mathcal{A}}(\gamma)P$. In other words, $\mathrm{M}_{\mathcal{A}}(\beta)$ is the image of $\mathrm{M}_{\mathcal{A}}(\gamma)$ under conjugation by P. $\square$

For a given algebra $\mathcal{A}$ and basis β we are free to use transformations in $\mathcal{R}_{\mathcal{A}}$ or matrices in $\mathrm{M}_{\mathcal{A}}(\beta)$ to capture the same structure. These isomorphisms are of fundamental importance to the study of $\mathcal{A}$-calculus. Notice, we say $(\mathcal{A}, \star)$ and $(\mathcal{B}, *)$ are **isomorphic** as real associative algebras and write $\mathcal{A} \approx \mathcal{B}$ if there exists an invertible $\mathbb{R}$-linear transformation $\Psi : \mathcal{A} \to \mathcal{B}$ such that $\Psi(x \star y) = \Psi(x) * \Psi(y)$ for all $x, y \in \mathcal{A}$.

Theorem 4.8. *If β is a basis for $\mathcal{A}$ then $\mathcal{A} \approx \mathcal{R}_{\mathcal{A}} \approx M_{\mathcal{A}}(\beta)$*

Proof: suppose $\beta = \{v_1, \dots, v_n\}$ is a basis for $\mathcal{A}$. Define $\Psi(\alpha) = \ell_\alpha$ as was studied in Equation 8. Notice $\Psi(c\alpha + \beta) = \ell_{c\alpha+\beta} = c\ell_\alpha + \ell_\beta = c\Psi(\alpha) + \Psi(\beta)$ hence Ψ is a linear transformation. Further, as $\mathcal{A}$ is unital there exists a multiplicative identity $\mathbb{1} \in \mathcal{A}$. If $T \in \mathcal{R}_{\mathcal{A}}$ then $T(\mathbb{1}) \in \mathcal{A}$ and for $x \in \mathcal{A}$ we calculate:

$$(\Psi(T(\mathbb{1})))(x) = \ell_{T(\mathbb{1})}(x) = T(\mathbb{1}) \star x = T(x). \tag{14}$$

Thus $\Psi(T(\mathbb{1})) = T$ which shows Ψ is a surjection. Suppose $\Psi(\alpha) = 0$ hence $\ell_\alpha(x) = 0$ for all $x \in \mathcal{A}$. Set $x = \mathbb{1}$ to calculate $\ell_\alpha(\mathbb{1}) = \alpha \star \mathbb{1} = \alpha = 0$. Therefore $\mathrm{Ker}(\Psi) = \{0\}$ and we find Ψ is an injection. Hence Ψ is an isomorphism of vector spaces. It remains to show Ψ preserves the algebra multiplication. We need associativity here:

$$\Psi(\alpha \star \beta)(x) = \ell_{\alpha\star\beta}(x) = (\alpha \star \beta) \star x = \alpha \star (\beta \star x) = \ell_\alpha(\ell_\beta(x)) = \ell_\alpha \circ \ell_\beta(x). \tag{15}$$

Therefore, Ψ gives the isomorphism $\mathcal{A} \approx \mathcal{R}_{\mathcal{A}}$. Next, define $\mathbf{M} : \mathcal{A} \to \mathrm{M}_{\mathcal{A}}(\beta)$ by

$$\mathbf{M}(\alpha) = [\ell_\alpha]_{\beta,\beta} = [[\alpha \star v_1]_\beta | \cdots | [\alpha \star v_n]_\beta]. \tag{16}$$

If $A \in \mathrm{M}_{\mathcal{A}}(\beta)$ then there exists $T \in \mathcal{R}_{\mathcal{A}}$ for which $[T]_{\beta,\beta} = A$. Use Equation 14 to see:

$$\mathbf{M}(T(\mathbb{1})) = [\ell_{T(\mathbb{1})}]_{\beta,\beta} = [T]_{\beta,\beta} = A. \tag{17}$$

Hence $\mathbf{M} : \mathcal{A} \to \mathrm{M}_{\mathcal{A}}(\beta)$ is surjective. Suppose $\mathbf{M}(\alpha) = 0$ then $[\ell_\alpha]_{\beta,\beta} = 0$ from which we find $\ell_\alpha = 0$. Thus, $\ell_\alpha(\mathbb{1}) = \alpha \star \mathbb{1} = \alpha = 0$. We find $\mathrm{Ker}(\mathbf{M}) = \{0\}$ and thus $\mathbf{M}$ is injective. Finally, we verify $\mathbf{M}$ preserves the algebra multiplication: we use the middle of Equation 15 in the second equality:

$$\mathbf{M}(x \star y) = [\ell_{x\star y}]_{\beta,\beta} = [\ell_x \circ \ell_y]_{\beta,\beta} = [\ell_x]_{\beta,\beta}[\ell_y]_{\beta,\beta} = \mathbf{M}(x)\mathbf{M}(y). \tag{18}$$

Thus $\mathbf{M}$ provides the isomorphism $\mathcal{A} \approx \mathrm{M}_{\mathcal{A}}(\beta)$. $\square$

It is convenient to set some notation for the inverse of the Ψ map. For each number $x \in \mathcal{A}$ the map Ψ assigns a linear transformation $\Psi(x) \in \mathcal{R}_{\mathcal{A}}$. It seems natural[4] to call the inverse of Ψ the **number map**. For convenience of notation, we also use # for the inverse of $\mathbf{M}$:

Definition 4.9. *The* **number map** $\# : \mathcal{R}_{\mathcal{A}} \to \mathcal{A}$ *is defined by* $\#(T) = T(\mathbb{1})$ *when the context demands. Likewise, given* β *a basis for* $\mathcal{A}$ *we define* $\# : M_{\mathcal{A}}(\beta) \to \mathcal{A}$ *and when the context demands* $\#([T]_{\beta,\beta}) = T(\mathbb{1})$.

The number map is easiest to understand when $\mathbb{1} = v_1 = e_1$ where $\mathcal{A} = \mathbb{R}^n$, however, we have taken care to allow for other possibilities[5].

Theorem 4.10. *If* $\beta = \{v_1, \dots, v_n\}$ *is a basis for* $\mathcal{A}$ *where* $v_1 = \mathbb{1}$ *then*

$$\mathbf{M}(x) = [[x]_\beta | [x \star v_2]_\beta | \cdots | [x \star v_n]_\beta]$$

and $\#(A) = \Phi_\beta^{-1}(\mathbf{col}_1(A))$ *where* $\Phi_\beta(x) = [x]_\beta$ *is the coordinate map.*

Proof: we define $\mathbf{M} : \mathcal{A} \to \mathrm{M}_{\mathcal{A}}(\beta)$ as in the proof of Theorem 4.8; $\mathbf{M}(x) = [\ell_x]_{\beta,\beta}$. The j-th column in $[\ell_x]_{\beta,\beta}$ is $[\ell_x(v_j)]_\beta = [x \star v_j]_\beta$. Therefore, as $v_1 = \mathbb{1}$ we find $\mathbf{M}(x) = [[x]_\beta | [x \star v_2]_\beta | \cdots | [x \star v_n]_\beta]$. Let $A = \mathbf{M}(x)$, we wish to solve for x. Notice, $\mathbf{col}_1(A) = [x]_\beta$ from which we derive $x = \Phi_\beta^{-1}(\mathbf{col}_1(A))$ hence $\#(A) = \Phi_\beta^{-1}(\mathbf{col}_1(A))$. $\square$

In almost all applications of the Theorem 4.10 we consider the case $\mathcal{A} = \mathbb{R}^n$ with $\beta = \{e_1, \dots, e_n\}$ the usual standard basis such that $e_1 = \mathbb{1}$. Given these special choices we obtain much improved formulae[6]

[4] this is **not** a hash-tag

[5] Note, $\mathbb{R} \times \mathbb{R}$ we have $\mathbb{1} = (1,1)$ and in $\mathbb{R}^{2\times 2}$ we have $\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ thus the usual standard bases for $\mathbb{R}^2$ or $\mathbb{R}^{2\times 2}$ do not include the identity of the algebra.

[6] when $\beta = \{e_1, \dots, e_n\}$ we drop β from the notation and simply write $\mathrm{M}_{\mathcal{A}}$ in the place of $\mathrm{M}_{\mathcal{A}}(\beta)$

Corollary 4.11. *If $\mathcal{A} = \mathbb{R}^n$ and $\mathbb{1} = e_1 = (1, 0, \dots, 0)$ then for $x \in \mathcal{A}$ and $A \in M_{\mathcal{A}}$,*

$$\mathrm{M}(x) = [x|x \star e_2|\cdots|x \star e_n] \qquad \& \qquad \#(A) = \mathbf{col}_1(A), \qquad \#\mathrm{M}(x) = x.$$

Notice that the first column of $A \in \mathrm{M}_{\mathcal{A}}$ determines the rest through the structure of the multiplication of $\mathcal{A}$. Setting aside the special context of the corollary, if $\beta = \{v_1, \dots, v_n\}$ is a non-standard basis with $v_j = \mathbb{1}$ then the j-th column of $\mathrm{M}(x)$ will fix the remaining columns according to the multiplication on $\mathcal{A}$. On the other hand, if $\mathbb{1} \notin \beta$ then there need not be a single column of each matrix in $\mathrm{M}_{\mathcal{A}}(\beta)$ which fixes the remaining columns. We see this phenomenon explicitly in Examples 4.26 and 4.31.

Definition 4.12. *We say $x \in \mathcal{A}$ is a* **unit** *if there exists $y \in \mathcal{A}$ for which $x \star y = y \star x = \mathbb{1}$. The set of all units is known as the* **group of units** *and we denote this by $\mathcal{A}^{\times}$. We say $a \in \mathcal{A}$ is a* **zero-divisor** *if $a \neq 0$ and there exists $b \neq 0$ for which $a \star b = 0$ or $b \star a = 0$. Let* $\mathbf{zd}(\mathcal{A}) = \{x \in \mathcal{A} \mid x = 0 \text{ or } x \text{ is a zero-divisor}\}$

Isomorphisms transfer both units and zero-divisors.

Theorem 4.13. *Suppose $(\mathcal{A}, \star)$ and $(\mathcal{B}, *)$ are real associative algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism. Then,*

(i.) $\Phi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$,

(ii.) *for $x \in \mathcal{A}^{\times}$, $\Phi(x^{-1}) = \Phi(x)^{-1}$; that is, $\Phi(\mathcal{A}^{\times}) = \mathcal{B}^{\times}$,*

(iii.) *if $x, y \in \mathcal{A}$ and $x \star y = 0$ then $\Phi(x) * \Phi(y) = 0$; that is, $\Phi(\mathbf{zd}(\mathcal{A})) = \mathbf{zd}(\mathcal{B})$.*

Proof: to prove (i.) simply note for each $y \in \mathcal{B}$ there exists $x \in \mathcal{A}$ for which $y = \Phi(x) = \Phi(x \star \mathbb{1}_{\mathcal{A}}) = \Phi(x) * \Phi(\mathbb{1}_{\mathcal{A}}) = y * \Phi(\mathbb{1}_{\mathcal{A}})$. Likewise, as $x = \mathbb{1}_{\mathcal{A}} \star x$ we have $y = \Phi(\mathbb{1}_{\mathcal{A}}) * y$. Thus, $\mathbb{1}_{\mathcal{B}} = \Phi(\mathbb{1}_{\mathcal{A}})$. As $\Phi(0) = 0$, the proofs of (ii.) and (iii.) are immediate from the definition of inverse and zero-divisor since $\Phi(x \star y) = \Phi(x) * \Phi(y)$. $\square$

Since $\mathcal{A} \approx \mathrm{M}_{\mathcal{A}}(\beta) \approx \mathcal{R}_{\mathcal{A}}$ we are free to focus our initial effort where is most convenient. When considering the characterization of $\mathbf{zd}(\mathcal{A})$ the representation $\mathrm{M}_{\mathcal{A}}(\beta)$ is useful due to the theory of determinants.

Theorem 4.14. *Let $M_{\mathcal{A}}(\beta)$ be the regular representation of $\mathcal{A}$ with respect to basis β. If $A \in M_{\mathcal{A}}(\beta)$ then either A is zero, a unit, or a zero-divisor.*

Proof: if $A \in \mathrm{M}_{\mathcal{A}}(\beta)$ then by definition there exists $S \in \mathcal{R}_{\mathcal{A}}$ for which $[S]_{\beta,\beta} = A$. Observe that either $\det(A) = 0$ or $\det(A) \neq 0$.

In the case $\det(A) \neq 0$ we know $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)^T$ in $\mathbb{R}^{n \times n}$. It remains[7] to show $A^{-1} \in \mathrm{M}_{\mathcal{A}}(\beta)$. Linear algebra provides the existence of $T : \mathcal{A} \to \mathcal{A}$ such that $[T]_{\beta,\beta} = A^{-1}$. Note,

$$[T]_{\beta,\beta}[S]_{\beta,\beta} = I \quad \Rightarrow \quad T \circ S = Id. \tag{19}$$

[7] in principle, you could worry the inverse exists in $\mathbb{R}^{n \times n}$ yet is not inside the regular representation of $\mathcal{A}$, the argument to follow shows this worry is needless.

Suppose $x, y \in \mathcal{A}$ and observe by right-$\mathcal{A}$-linearity of S we derive:

$$T(S(x) \star y) = T(S(x \star y)) = x \star y = T(S(x)) \star y. \tag{20}$$

Since S is a surjection the calculation above shows T is right-$\mathcal{A}$-linear hence $[T]_{\beta,\beta} = A^{-1} \in \mathrm{M}_{\mathcal{A}}(\beta)$. Thus A is a unit in $\mathrm{M}_{\mathcal{A}}(\beta)$.

On the other hand, if $\det(A) = 0$ then the constant term in the minimal polynomial $m(t)$ of A is zero. Thus, either $A = 0$ or there exists some $k \geq 2$ for which $m(t) = t^k + c_{k-1}t^{k-1} + \cdots + c_1 t$ for some $c_1, \cdots, c_{k-1} \in \mathbb{R}$. From the theory of the minimal polynomial we know $m(A) = 0$ hence as A factors either to the left or right:

$$0 = m(A) = A\left(A^{k-1} + c_{k-1}A^{k-2} + \cdots + c_1 I\right) = \left(A^{k-1} + c_{k-1}A^{k-2} + \cdots + c_1 I\right)A. \tag{21}$$

Thus $B = A^{k-1} + c_{k-1}A^{k-2} + \cdots + c_1 I$ gives $AB = 0 = BA$. Furthermore, since $\mathrm{M}_{\mathcal{A}}(\beta)$ is an algebra and $A \in \mathrm{M}_{\mathcal{A}}(\beta)$ it is clear that $B \in \mathrm{M}_{\mathcal{A}}(\beta)$. Thus in the case $\det(A) = 0$ either $A = 0$ or A is a zero-divisor. $\square$

The minimal polynomial argument used in the zero-divisor case could also have been used to show the inverse of a unit A in $\mathrm{M}_{\mathcal{A}}(\beta)$ is formed from a suitable polynomial in A.

Corollary 4.15. *Every element of $\mathcal{A}$ and $\mathcal{R}_{\mathcal{A}}$ is either zero, a unit, or a zero-divisor.*

Proof: combine Theorem 4.13 and Theorem 4.14. $\square$

We study finite dimensional real associative algebras in this work. If $\dim(\mathcal{A}) = n$ then geometrically $\mathcal{A}$ is essentially just $\mathbb{R}^n$. For $\mathrm{M}_{\mathcal{A}}(\beta)$ we have an n-dimensional subspace of $\mathbb{R}^{n\times n}$. The set $\mathbf{zd}(\mathrm{M}_{\mathcal{A}}(\beta))$ is the solution set of $\det(A) = 0$. This is an n-th order polynomial equation in the components of A which includes $A = 0$ and at most an $(n-1)$-dimensional space of zero-divisors. For example, if we consider the complex numbers $\mathcal{A} = \mathbb{C}$ then $\mathbf{zd}(\mathcal{A}) = \{0\}$. On the other hand $\mathbb{R}^2$ with the standard direct product given by $(a, b) \star (c, d) = (ac, bd)$ has $\mathbf{zd}(\mathbb{R}^2) = (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$. We observe that $\mathbf{zd}(\mathcal{A})$ is formed by a union of subspaces of $\mathcal{A}$. This is natural given the following:

Theorem 4.16. *The set $\mathbf{zd}(\mathcal{A})$ is fixed under negation.*

Proof: suppose $x \in \mathbf{zd}(\mathcal{A})$ then there exists $y \in \mathcal{A}$ for which $x \star y = 0$. Thus $-x \star y = 0$ and we find $-x \in \mathbf{zd}(\mathcal{A})$. $\square$

Theorem 4.17. *Let $\mathcal{A}$ be an n-dimensional real associative unital algebra. There exists an* **invertible basis** $\beta = \{v_1, \dots, v_n\} \subset \mathcal{A}^{\times}$.

Proof: Let $\gamma = \{w_1, \dots, w_n\}$ be a basis for $\mathcal{A}$. Suppose that a basis element w_i is a zero divisor. Recall that the zero divisors in an n dimensional algebra can be at most $n-1$ dimensional. Hence, there exist $c_1, \dots, c_n$ with $c_i \neq 0$ (since this restriction only removes another $n-1$ dimensional subspace of the algebra) such that $\{w_1, \dots, c_1 w_1 + \cdots + c_i w_i + \cdots + c_n w_n, \dots, w_n\}$ is a basis for $\mathcal{A}$ where the i-th component is now a unit, since the transformation $w_i \mapsto c_1 w_1 + \cdots + c_i w_i + \cdots + c_n w_n$ preserves linear independence of the basis so long as

$c_i \neq 0$, and we know by linear algebra that the span of the vectors must also be preserved, since we have n linearly independent vectors in an n dimensional vector space. Therefore, applying this procedure iteratively to $w_1, w_2, \dots, w_n$ yields a basis $\beta = \{v_1, v_2, \dots, v_n\}$ for $\mathcal{A}$ where $\beta^* \subset \mathcal{A}^\times$ $\square$

It is easy to see the argument above also allows the following result:

Corollary 4.18. *Let $\mathcal{A}$ be an n-dimensional real associative unital algebra. There exists an invertible basis of the special form $\beta = \{\mathbb{1}, v_2, \dots, v_n\} \subset \mathcal{A}^\times$.*

Finally, we make the following observation that any open ball about a point in $\mathcal{A}$ necessarily intersects infinitely many points in $\mathcal{A}^\times$. This observation should be geometrically evident since $\mathbf{zd}(\mathcal{A})$ is a space of smaller dimension than $\mathcal{A}$ and $\mathcal{A} - \mathbf{zd}(\mathcal{A}) = \mathcal{A}^\times$.

Theorem 4.19. *Let $\mathcal{A}$ be an n-dimensional real associative unital algebra. The group of units $\mathcal{A}^\times$ is a dense subset of $\mathcal{A}$.*

4.1 Examples

To explain the structure of complex numbers it suffices to say $i^2 = -1$ and then just add and multiply $a + bi, c + di$ as usual. Of course, we can be more explicit in our construction if the audience knows about field extensions or group algebras, but, as a starting point it is convenient to provide definitions of algebras which are accessible to every level of student.

Example 4.20. *The* **real numbers** *with their usual addition and multiplication is an associative algebra over $\mathbb{R}$. If $a \in \mathbb{R}$ then $[a] \in M_{\mathbb{R}} = \mathbb{R}^{1\times 1}$ is its left regular representation. Usually we will not distinguish between a and $[a]$.*

Example 4.21. *The* **complex numbers** *are defined by $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ where $i^2 = -1$. If $a + ib, c + id \in \mathbb{C}$ then $(a + ib)(c + id) = ac + iad + ibc + i^2bd = ac - bd + i(ad + bc)$. Note every nonzero complex number $a + ib$ has multiplicative inverse $\frac{a-ib}{a^2+b^2}$ hence $\mathbb{C}$ is a field. Note $\mathrm{M}(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and the set of all such matrices is denoted $M_{\mathbb{C}}$.*

Example 4.22. *The* **hyperbolic** *numbers are given by $\mathcal{H} = \mathbb{R} \oplus j\mathbb{R}$ where $j^2 = 1$. If $a + jb, c + jd \in \mathcal{H}$ then $(a + jb)(c + jd) = ac + adj + jbc + j^2bd = ac + bd + j(ad + bc)$. Observe $\mathrm{M}(a+bj) = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in M_{\mathcal{H}}$ and $\mathbf{zd}(\mathcal{H}) = \{a+bj \mid a^2 = b^2\}$ whereas $\mathcal{H}^\times = \{a+bj \mid a^2 \neq b^2\}$. The reciprocal of an element in $\mathcal{H}^\times$ is simply*

$$\frac{1}{a + bj} = \frac{a - bj}{a^2 - b^2} \tag{22}$$

this follows from the identity $(a + bj)(a - bj) = a^2 - b^2$ given $a^2 - b^2 \neq 0$. Let $\mathcal{B} = \mathbb{R} \times \mathbb{R}$ with $(a, b)(c, d) = (ac, bd)$ for all $(a, b), (c, d) \in \mathcal{B}$. We can show that

$$\Psi(a, b) = a\left(\frac{1 + j}{2}\right) + b\left(\frac{1 - j}{2}\right) \qquad \& \qquad \Psi^{-1}(x + jy) = (x + y, x - y) \tag{23}$$

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provide an isomorphism of $\mathcal{H}$ and $\mathbb{R}\times\mathbb{R}$. We can use this isomorphism to transfer problems from $\mathcal{H}$ to $\mathcal{B}$ and vice-versa. For example, to solve $z^2+Bz+C=0$ in the hyperbolic numbers we note

$$z^2+Bz+C=0 \quad\Rightarrow\quad \Psi^{-1}(z)^2+\Psi^{-1}(B)\Psi^{-1}(z)+\Psi^{-1}(C)=0 \tag{24}$$

Setting $\Psi^{-1}(B)=(b_1,b_2)$ and $\Psi^{-1}(C)=(c_1,c_2)$ and $\Psi^{-1}(z)=(x,y)$ we arrive at

$$(x,y)^2+(b_1,b_2)(x,y)+(c_1,c_2)=0 \tag{25}$$

which reduces to

$$(x^2+b_1x+c_1, y^2+b_2y+c_2)=(0,0). \tag{26}$$

Of course, these are just quadratic equations in $\mathbb{R}$ so we can solve them and transfer back the result to the general solution of $z^2+Bz+C=0$ in $\mathcal{H}$. Given this correspondence, we deduce there are either zero, two or four solutions to the quadratic hyperbolic equation.

Example 4.23. *The* **dual numbers** *are given by $\mathcal{N}=\mathbb{R}\oplus\epsilon\mathbb{R}$ where $\epsilon^2=0$. If $a+\epsilon b, c+\epsilon d\in\mathcal{N}$ then*

$$(a+\epsilon b)(c+\epsilon d)=ac+ad\epsilon+bc\epsilon+\epsilon^2bd=ac+(ad+bc)\epsilon. \tag{27}$$

Observe $\mathbf{M}(a+b\epsilon)=\begin{bmatrix} a & 0\\ b & a\end{bmatrix}\in M_{\mathcal{N}}$ and $\mathbf{zd}(\mathcal{N})=\{a+b\epsilon \mid a^2=0\}=\epsilon\mathbb{R}$. The units in the dual numbers are of the form $a+b\epsilon$ where $a\neq 0$. Note $(a+b\epsilon)(a-b\epsilon)=a^2$ hence $\frac{1}{a+b\epsilon}=\frac{a-b\epsilon}{a^2}$ provided $a\neq 0$.

For higher dimensional algebras the multiplicative inverse of a general element can be calculated by computing the inverse of the element's regular representation.

Example 4.24. *The* **n-th order dual numbers** *are given by $\mathcal{N}_n=\mathbb{R}\oplus\eta\mathbb{R}\oplus\cdots\oplus\epsilon^{n-1}\mathbb{R}$ where $\epsilon^n=0$ and $\epsilon^k\neq 0$ for $1\leq k\leq n-1$. The regular representation is formed by lower triangular matrices of a particular type:*

$$\mathbf{M}(a_1+a_2\epsilon+\cdots+a_n\epsilon^{n-1})=\begin{bmatrix} a_1 & 0 & \cdots & 0 & 0\\ a_2 & a_1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a_{n-1} & a_{n-2} & \cdots & a_1 & 0\\ a_n & a_{n-1} & \cdots & a_2 & a_1\end{bmatrix}\in M_{\mathcal{N}_n} \tag{28}$$

Notice $a_1+a_2\epsilon+\cdots+a_n\epsilon^{n-1}\in\mathbf{zd}(\mathcal{N}_n)$ only if $a_1\neq 0$.

Example 4.25. *Let $\mathcal{A}=\mathbb{R}\oplus j\mathbb{R}\oplus j^2\mathbb{R}$ where $j^3=1$. The matrix representatives of these numbers have an interesting shape; note: $A\in M_{\mathcal{A}}$ implies $A=\begin{bmatrix} a & c & b\\ b & a & c\\ c & b & a\end{bmatrix}$. We note an isomorphism $\mathcal{A}\approx\mathbb{R}\times\mathbb{C}$ is given by mapping j to $(1,\omega)$ where ω is a third root of unity.*

Example 4.26. *Let $\mathcal{A}=\mathbb{R}\times\mathcal{H}$ where $\mathbb{1}=(1,1+0j)$. Let $\beta=\{(1,0),(0,1),(0,j)\}$ gives block-diagonal $A\in M_{\mathcal{A}}(\beta)$;*

$$\mathbf{M}_\beta((a,b+cj))=\begin{bmatrix} a & 0 & 0\\ 0 & b & c\\ 0 & c & b\end{bmatrix}. \tag{29}$$

This algebra is isomorphic to $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ with $(a_1,a_2,a_3)\star(b_1,b_2,b_3)=(a_1b_1,a_2b_2,a_3b_3)$.

Example 4.27. *Let $\mathcal{A} = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R} \oplus j^3\mathbb{R}$ where $j^4 = 1$. Observe,*

$$\mathrm{M}(a + bj + cj^2 + dj^3) = \begin{bmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{bmatrix}. \tag{30}$$

This algebra is naturally isomorphic to $\mathbb{C} \oplus \mathcal{H}$ which is clearly isomorphic to $\mathbb{C} \times \mathbb{R} \times \mathbb{R}$.

Example 4.28. *Let $\mathcal{A} = \mathcal{H} \times \mathcal{H}$ where $\mathbb{1} = (1 + 0j, 1 + 0j)$. This means $(1, 1)$ is naturally represented by the identity matrix. Set $\beta = \{(1, 0), (j, 0), (0, 1), (0, j)\}$ and observe*

$$\mathrm{M}_\beta((a + bj, c + dj)) = \left[\begin{array}{cc|cc} a & b & 0 & 0 \\ b & a & 0 & 0 \\ \hline 0 & 0 & c & d \\ 0 & 0 & d & c \end{array}\right]. \tag{31}$$

This algebra is isomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with the Hadamard product $(a_1, a_2, a_3, a_4) * (b_1, b_2, b_3, b_4) = (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$.

Example 4.29. *Let $\mathcal{A} = \mathbb{C} \times \mathbb{C}$ where $\mathbb{1} = (1 + 0i, 1 + 0i)$. Here we study the problem of two complex variables. In this algebra $(1 + 0i, 1 + 0i)$ corresponds to the identity and hence $(1, 1)$ is naturally represented by the identity matrix. In total we have once more a block-diagonal representation: $A \in M_{\mathcal{A}}$ implies $A = \left[\begin{array}{cc|cc} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ \hline 0 & 0 & c & -d \\ 0 & 0 & d & c \end{array}\right]$ and this matrix represents $(a + bi, c + di)$.*

Example 4.30. *Let $\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$ where $i^2 = j^2 = k^2 = -1$ and $ij = k$. These are Hamilton's famed* **quaternions**. *We can show $ij = -ji$ hence these are not commutative. With respect to the natural basis $e_1 = 1, e_2 = i, e_3 = j, e_4 = k$ we find the matrix representative of $a + ib + cj + dk$ is as follows:*

$$A = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \in M_{\mathbb{H}}. \tag{32}$$

Example 4.31. *Let $\mathcal{A} = \mathbb{R}_2$ with the multiplication $\star$ induced from the multiplication of 2×2 matrices. This again forms a noncommutative algebra. In particular, this multiplication is induced in the natural manner:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t & x \\ y & z \end{bmatrix} = \begin{bmatrix} at + by & ax + bz \\ ct + dy & cx + dz \end{bmatrix}. \tag{33}$$

It follows that $(a, b, c, d) \star (t, x, y, z) = (at + by, ax + bz, ct + dy, cx + dz)$. We can read from this multiplication that the representative of $(a, b, c, d) \in \mathbb{R}_2$ is given by

$$A = \left[\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & a & 0 & b \\ \hline c & 0 & d & 0 \\ 0 & c & 0 & d \end{array}\right] = \left[\begin{array}{c|c} aI & bI \\ \hline cI & dI \end{array}\right] \in M_{\mathcal{A}}. \tag{34}$$

Note, the basis $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ *does not contain the multiplicative identity* $I = E_{11} + E_{22}$. *We define* $\mathcal{B} = \mathbb{R}^4$ *by*

$$(a, b, c, d) \star (w, x, y, z) = (aw + by, ax + bz, cw + dy, cx + dz) \tag{35}$$

Of course, $(1, 0, 1, 0) = \mathbb{1}_{\mathcal{B}}$ *and* $\mathcal{B}$ *is really just another notation for the* 2×2 *matrix algebra. In fact,* $\Psi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a, b, c, d)$ *defines an isomorphism of* $\mathcal{A}$ *and* $\mathcal{B}$. *The reader will verify that* $\Psi(AB) = \Psi(A) \star \Psi(B)$.

Example 4.32. Z *in* $\mathbb{H}^{2\times 2}$ *is a* $\mathcal{A}$*-number. There is a natural injection* $\Psi : \mathbb{H}^{2\times 2} \to \mathbb{R}^{8\times 8}$ *induced from* $\mathrm{M} : \mathbb{H} \to \mathbb{R}^4$ *from Example 4.30,*

$$\Psi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = \begin{bmatrix} \mathrm{M}(x) & \mathrm{M}(y) \\ \mathrm{M}(z) & \mathrm{M}(w) \end{bmatrix} \tag{36}$$

for all $x, y, z, w \in \mathbb{H}$. *The matrices in* $\Psi(\mathbb{H}^{2\times 2})$ *are isomorphic to* $\mathbb{H}^{2\times 2}$.

Example 4.33. *Let* G *be a finite multiplicative group;* $G = \{g_1, \dots, g_n\}$ *then we define*

$$\mathcal{A}_G = g_1\mathbb{R} \oplus \cdots \oplus g_n\mathbb{R} \tag{37}$$

with natural multiplication inherited from G. *For example, for all* $a, b, c, d \in \mathbb{R}$,

$$(ag_1 + bg_2)(cg_3 + dg_4) = acg_1g_3 + adg_1g_4 + bcg_2g_3 + bdg_2g_4. \tag{38}$$

The group algebra allows us to multiply $\mathbb{R}$*-linear combinations of group elements by extending the group multiplication linearly. By construction,* $\{g_1, \dots, g_n\}$ *serves as a basis for* $\mathcal{A}_G$. *As* G *is a group we know for each* $i, j \in \{1, \dots, n\}$ *there exists* $k \in \{1, \dots, n\}$ *for which* $g_ig_j = g_k$. *If we define structure constants* C_{ijk} *by* $g_ig_j = \sum_l C_{ijl}g_l$ *then* $g_ig_j = g_k$ *implies* $C_{ijl} = \delta_{kl}$.

Example 4.34. *The cyclic group of order* n *in multiplicative notation has the form* $G = \{e, g, g^2, \dots, g^{n-1}\}$. *The group algebra* $\mathcal{A}_G = e\mathbb{R} \oplus g\mathbb{R} \oplus \cdots \oplus g^{n-1}\mathbb{R}$. *We usually call this algebra the* n*-hyperbolic numbers.*

5 $\mathcal{A}$-differentiable functions

Given $\mathcal{A}$ with basis $\beta = \{\mathbb{1}, v_2, \dots, v_n\}$ we may define an inner-product on $\mathcal{A}$ by bilinearly extending $g(v_i, v_j) = \delta_{ij}$. The g-induced norm $||x|| = \sqrt{g(x, x)}$ has $||v_i|| = 1$ for $i = 1, 2, \dots, n$. In this construction we have β is g-orthonormal and

$$||x_1v_1 + x_2v_2 + \cdots + x_nv_n||^2 = x_1^2 + x_2^2 + \cdots + x_n^2. \tag{39}$$

hence $||\zeta|| = ||\overline{\zeta}_j||$ for $j = 2, \dots, n$.

We should caution, there are examples where the length of $\mathbb{1}$ is not 1 in the natural norm for the example. For example, $\mathbb{R}^n$ with $\mathbb{1} = (1, \dots, 1)$ has length $||\mathbb{1}|| = \sqrt{n}$ in the usual