Your PRINTED NAME indicates you read Chapter 5 of the notes: ____________________.

Problem 91 For the inner product space $(V, \langle , \rangle)$ and dual vector $\alpha \in \mathcal{L}(V, \mathbb{F})$ find $y \in V$ for which $\alpha(x) = \langle x, y \rangle$ for each $x \in V$,

(a.) $\alpha(x) = x_1 - 2x_2 + 4x_3$ for $x \in V = \mathbb{R}^3$ has $\langle x, y \rangle = x^T y$ for $x, y \in \mathbb{R}^3$

(b.) $\alpha(z) = iz_1 - (3+i)z_2$ for $z \in V = \mathbb{C}^2$ has $\langle z, w \rangle = z_1 w_1^* + z_2 w_2^*$ for $z, w \in \mathbb{C}^2$

(c.) $\alpha(f) = f(0) + f'(1)$ for $f \in V = P_2(\mathbb{R})$ has $\langle f, g \rangle = \displaystyle\int_0^1 f(t)g(t)\,dt$

Problem 92 The Riesz Representation Theorem in a finite dimensional inner product space $(V, \langle , \rangle)$ states that each linear functional $\phi : V \to \mathbb{F}$ has the form $\phi(x) = \langle x, z \rangle$ for a unique $z \in V$. We call z the **Riesz vector of** ϕ and denote $z = \sharp\phi$. Conversely, we denote $\flat z = \phi$ in this case. The maps $\sharp : V^* \to V$ and $\flat : V \to V^*$ are sometimes called the **musical morphisms** as they provide natural isomorphisms between and inner product space[1] and its dual space.

(a.) Consider $\phi(x, y, z) = 3x - y + z$. Find $\sharp\phi$ for $\mathbb{R}^3$ with the dot-product,

(b.) Consider $\phi(A) = A_{11} + A_{21}$ for $A \in \mathbb{R}^{2\times 2}$ find $\sharp\phi$.

Problem 93 Consider $\alpha \in V^*$ where $(V, \langle \;,\; \rangle)$ is an inner product space. Then we define $\sharp\alpha$ to be the unique vector for which $\alpha(x) = \langle x, \sharp\alpha \rangle$. The vector $\sharp\alpha$ is sometimes called the **Riesz' vector** of α. The maps $\sharp : V^* \to V$ and its inverse $\flat : V \to V^*$ are sometimes called the **musical morphisms**. They provide natural isomorphisms between V and V^*. Calculate $\sharp\alpha$ for the following:

(a.) $V = \mathbb{R}^3$, $\alpha(x) = x_1 + 2x_2 + 3x_3$

(b.) $V = \mathbb{C}^2$, $\alpha(z) = z_1 + iz_2$

(c.) $V = P_2(\mathbb{R})$ with $\langle f(x), h(x) \rangle = \int_0^1 f(t)h(t)\,dt$ given $\alpha(f) = f(0) + f'(1)$.

(d.) $V = \mathbb{R}^{2\times 2}$ with $\langle A, B \rangle = tr(AB^T)$ given $\alpha(A) = A_{11} + A_{21}$.

Remark: *we should see in class at some day, the restriction of an inner product is once more an inner product. If we study natural generalizations of inner products then we find this restriction property fails. That is the point of the next problem.*

Problem 94 A **scalar product** or **metric** on $\mathbb{R}^4$ is a symmetric bilinear form which is **nondegenerate**. In particular, $g : V \times V \to \mathbb{R}$ is **nondegenerate** if $g(v, w) = 0$ for all $w \in V$ implies $v = 0$. Nondegeneracy is equivalent to the condition that the matrix of g has nonzero determinant. It can be shown that every inner product is a metric, however, the converse

[1] actually, this construction still makes sense for a metric space, but, I'm focusing on inner product spaces for your course

fails. This problem intends to illustrate some of the differences. The metric given below is the so-called **Minkowski Metric** of time-space. Let $g : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be defined by:

$$g(v, w) = v^T \eta w \qquad \text{where} \qquad \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a.) show g is a symmetric bilinear form on $\mathbb{R}^4$

(b.) why is g is **not** an inner product on $\mathbb{R}^4$?

(c.) let $W = \text{span}\{e_2, e_3, e_4\}$ and show $g|_W$ is an inner product (and hence a metric).

(d.) let $C = \text{span}\{e_2 - e_1\}$ and show $g|_C$ is **not** an metric.

Problem 95 Suppose (V, g) is a real geometry meaning that V is a real vector space paired with a bilinear, symmetric, nondegenerate form g. Given basis $\Upsilon = \{v_1, \dots, v_n\}$ for V and $\alpha \in V^*$ we define $\sharp\alpha \in V$ by $\alpha(x) = g(x, \sharp\alpha)$ for each $x \in V$.

(a.) Prove $\sharp : V^* \to V$ defines an isomorphism.

(b.) If $\flat = \sharp^{-1}$ then find the explicit formula for $\flat(x)$ when $x = \sum_{i=1}^n x^i v_i$

(c.) Show (V^*, g^*) is also a real geometry given we define $g^*(\alpha, \beta) = g(\sharp\alpha, \sharp\beta)$.

Problem 96 (**Gwyneth's Musical Morphism Problem**) Let $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a metric with matrix $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. If $v = \sum_{i=1}^3 v^i e_i = (a, b, c)$ then calculate v_i for $i = 1, 2, 3$. Also, show $g(v, v) = \sum_{i=1}^3 v^i v_i$ (you can show it in terms of a, b, c or v^1, v^2, v^3 whatever you prefer)

Problem 97 Let $T : V \times V \times V^* \to \mathbb{R}$ be a multilinear mapping. Determine if $S = T \circ (\sharp, \sharp, \flat)$ is also a multilinear mapping. Also, find the coordinate transformation rules for T and S. Here we assume (V, g) is a real geometry.

Problem 98 Let V be a real vector space and $x, y \in V$. Define $x \otimes y : V^* \times V^* \to \mathbb{R}$ according to the rule $(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$. Show $x \otimes y$ is a bilinear mapping on $V^* \times V^*$. If V has basis $\beta = \{v_1, \dots, v_n\}$ show $\Upsilon = \{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ serves as a basis for $\mathcal{B}(V^*)$ which is the set of all real-valued bilinear maps on V^*.

Problem 99 Let $F : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ be defined by the determinant:

$$F(v_1, \dots, v_n) = det(v_1 | \cdots | v_n)$$

The standard basis $\{e_1, \dots, e_n\}$ has dual basis $\{e^1, \dots, e^n\}$ for which $e^i(e_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. Any k-linear form on $\mathbb{R}^n$ can be expressed as a linear combination of the basic multilinear maps $e^{i_1} \otimes \cdots \otimes e^{i_k}$.

(a.) Express $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as explicit sum of the tensor product of e^1, e^2.

(b.) Express $F : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ as explicit sum of the tensor product of e^1, e^2, e^3.

(c.) Express $F : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ as an explicit sum of the tensor product of $e^1, \dots, e^n$ with the help of the beautifully elegant notation $\epsilon_{i_1 \dots i_n}$.

(d.) Let $G : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ be a completely antisymmetric n-linear map for which $G(e_1, \dots, e_n) = 1$. Prove $F = G$.

Problem 100 **The cross product:** given $(n-1)$-vectors in $\mathbb{R}^n$ we know that if these are LI then there exists a unique line which is perpendicular to the span of the given vectors. It follows there are exactly two unit-vectors which are orthogonal to the $(n-1)$-dimensional subspace. There is a slick formula to generate a standard choice of one of those unit-vectors if we are given $(n-1)$-orthonormal vectors. In this problem we explore this formula, which you've already seen in $n = 3$, but perhaps the other cases are novel. Notice here T defines an $\mathbb{R}^n$-valued $(n-1)$-linear mapping on $\mathbb{R}$. Most examples we consider in this course of multilinear maps are simply real-valued so this problem expands our view, yet, also connects with something we've already studied.

(a.) For all $a, b \in \mathbb{R}^2$ we define

$$T(a) = \sum_{j=1}^{2} \big(\det[a|e_j]\big) e_j.$$

Show: $a \bullet T(a) = 0$ and $T(a) \bullet b = \det[a|b]$. Show $\|a\| = 1$ implies $\|T(a)\| = 1$.

(b.) For all $a, b, c \in \mathbb{R}^3$ we define

$$T(a, b) = \sum_{j=1}^{3} \big(\det[a|b|e_j]\big) e_j.$$

Show: $T(a, b) \bullet c = \det[a|b|c]$ and show $\{a, b\}$ orthonormal implies $\{a, b, c\}$ is orthonormal.

(c.) For all $a, b, c \in \mathbb{R}^4$ we define

$$T(a, b, c) = \sum_{j=1}^{4} \big(\det[a|b|c|e_j]\big) e_j$$

Show: $T(a, b, c) \bullet d = \det[a|b|c|d]$ and If $\{a, b, c\}$ is orthonormal then $\{a, b, c, T(a, b, c)\}$ is orthonormal.