Your PRINTED NAME indicates you read Chapter 6 of the notes: ____________________.

Problem 101 Suppose V is a vector space of finite dimension over $\mathbb{F}$ a field[1]. Let $v_1, v_2, \dots, v_k \in V$ then we say $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ is a **k-vector** and the span of all such k-vectors forms $\Lambda^k(V)$. We assume the wedge product is k-linear in the sense that:

$$v_1 \wedge \cdots \wedge v_j + cw_j \wedge \cdots \wedge v_k = v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k + c\, v_1 \wedge \cdots \wedge w_j \wedge \cdots \wedge v_k$$

for $j = 1, 2, \dots, k$. Furthermore, we assume the wedge product is antisymmetric with respect to the exchange of any adjacent pair of vectors:

$$v_1 \wedge \cdots \wedge v_j \wedge v_{j+1} \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_{j+1} \wedge v_j \wedge \cdots \wedge v_k.$$

Finally, we assume the wedge product is associative.

(a.) Show that if $k > n$ then $v_1 \wedge \cdots \wedge v_k = 0$.

(b.) Let $k \leq n$. Show that if $v_1, \dots, v_n$ are LI then the set of k-vectors of the form $v_{i_1} \wedge \cdots \wedge v_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ is a LI set.

(c.) Let $k \leq n$. Show $dim(\Lambda^k(V)) = \binom{n}{k}$.

(d.) Let $\Omega(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V)$ where $\Lambda^0 = span\{1\} = \mathbb{F}$. Show Ω is a graded commutative algebra in that $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$ implies $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ where $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

(e.) Suppose $v_1, \dots, v_k$ is a linearly dependent set of vectors. Prove $v_1 \wedge \cdots \wedge v_k = 0$.

Problem 102 In $\mathbb{R}^3$ the exterior algebra of differential forms gives us an alternative method to study vector calculus. In particular, 0-forms are real-valued functions of $\mathbb{R}^3$ whereas 1-forms and 2-forms correspond to vector fields

$$\omega_{\vec{A}} = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 \;\;\&\;\; \Phi_{\vec{B}} = B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2.$$

Every 3-form has the form $g dx_1 \wedge dx_2 \wedge dx_3$ where g is a real-valued function on $\mathbb{R}^3$.

(a.) Show $\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A} \times \vec{B}}$.

(b.) Show $\omega_{\vec{A}} \wedge \omega_{\vec{B}} \wedge \omega_{\vec{C}} = (\,(\vec{A} \times \vec{B}) \bullet \vec{C}\,)\, dx_1 \wedge dx_2 \wedge dx_3$.

Remark: there is much more to say here, but I'll stop here since these suffice to show the wedge product creates both cross and dot-products for vector fields. Furthermore, since we know the triple product identity $(\vec{A} \times \vec{B}) \bullet \vec{C} = det(\vec{A}|\vec{B}|\vec{C})$ we find a natural transition to the wedge product definition of the determinant:

Let $A \in \mathbb{F}^{n \times n}$ then $det(A)$ is the unique scalar defined implicitly by:

$$\boxed{col_1(A) \wedge \cdots \wedge col_n(A) = det(A) e_1 \wedge \cdots \wedge e_n.}$$

[1] not of characteristic 2 hence $2 \neq 0$ and $-x = x$ implies $x = 0$

Problem 103 In view of the above definition, calculate $det(A)$ for the following matrices:

(a.) $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 9 \\ 3 & 6 & 8 & 8 \\ 4 & 8 & 3 & 4 \end{bmatrix}$,

(b.) $A = I \in \mathbb{R}^{n \times n}$, the $n \times n$ identity matrix.

(c.) $A = M \oplus N$ where $M = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 5 & 5 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

Problem 104 If $T : V \to V$ is a linear transformation then it naturally induces a **p-th exterior power** denoted $\Lambda^p T : \Lambda^p V \to \Lambda^p V$ by $\Lambda^p T(v_1 \wedge \cdots \wedge v_p) = T(v_1) \wedge \cdots \wedge T(v_p)$. Explain:

(a.) if $S, T : V \to V$ then $\Lambda^p(S \circ T) = \Lambda^p S \circ \Lambda^p T$

(b.) $(\Lambda^{p+q}T)\,(\mu \wedge \gamma) = (\Lambda^p T)\,(\mu) \wedge (\Lambda^q T)\,(\mu)$

(c.) if T_{JI} is matrix of the p-th exterior power of T in the sense that $(\Lambda^p T)\,(e_I) = \sum_J e_J T_{JI}$ then calculate the $p = 2$ matrix $[T_{JI}]$ in the case V has basis $\beta = \{e_1, e_2, e_3\}$. Note, the natural lexographic ordered $\beta^2 = \{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$.

(d.) Since $\Lambda^3(T)(e_1 \wedge e_2 \wedge e_3) = \det(T) e_1 \wedge e_2 \wedge e_3$ we can use (b.) to uncover a myriad of formulas for the determinant. For example, as we define $\Lambda^1 T = T$. Write out

$$\Lambda^{2+1}(T)(e_1 \wedge e_2 \wedge e_3) = T(e_1) \wedge \Lambda^2(T)(e_1 \wedge e_2)$$

and identify the identity as a particular **cofactor expansion** about a row or column.

Problem 105 Let $A \in \mathbb{F}^{n \times n}$ and suppose $v_1, \dots, v_k \in \mathbb{F}^n$ then we define

$$A(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$$

for each $v_1 \wedge \cdots \wedge v_k \in \Lambda^k(\mathbb{F}^n)$. Given $A, B \in \mathbb{F}^{n \times n}$, and $\gamma \in \Lambda^k(\mathbb{F}^n)$, show the product above satisfies:

$$(AB)\gamma = A(B\gamma).$$

Consider $\gamma = e_1 \wedge \cdots \wedge e_n$ and show that $det(AB) = det(A)det(B)$ follows. Also, use the exterior algebra to derive the identity $det(cA) = c^n det(A)$ for $A \in \mathbb{F}^{n \times n}$ and $c \in \mathbb{F}$.

Problem 106 **A matrix model of the exterior algebra on $\mathbb{R}^2$:**
Consider $\beta = \{1, e_1, e_2, e_1 \wedge e_2\}$ serve to generate $V = \text{span}(\beta)$ as a real vector space of dimension 4. I'll arrange the $\wedge$ products in a table:

$\wedge$	1	e_1	e_2	$e_1 \wedge e_2$
1	1	e_1	e_2	$e_1 \wedge e_2$
e_1	e_1	0	$e_1 \wedge e_2$	0
e_2	e_2	$-e_1 \wedge e_2$	0	0
$e_1 \wedge e_2$	$e_1 \wedge e_2$	0	0	0

Linear transformations on V naturally correspond to 4×4 matrices. Notice, we can define the left multiplication maps with respect to the wedge product: $L_x(y) = x \wedge y$. Then, the set of $[L_x]_{\beta,\beta} \in \mathbb{R}^{4\times 4}$. I'll set it up:

$$\begin{aligned}[L_x]_{\beta,\beta} &= [[L_x(1)]_\beta | [L_x(e_1)]_\beta | [L_x(e_2)]_\beta | [L_x(e_1 \wedge e_2)]_\beta] \\ &= [[x]_\beta | [x \wedge e_1]_\beta | [x \wedge e_2]_\beta | [x \wedge e_1 \wedge e_2]_\beta]\end{aligned}$$

For example,

$$[L_{e_1}]_{\beta,\beta} = [[e_1]_\beta | [e_1 \wedge e_1]_\beta | [e_1 \wedge e_2]_\beta | [e_2 \wedge e_1 \wedge e_2]_\beta] = [[e_1]_\beta | 0 | [e_1 \wedge e_2]_\beta | 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a.) Complete my work by finding $[L_{e_2}]_{\beta,\beta}$, $[L_{e_1 \wedge e_2}]_{\beta,\beta}$ and the far more relaxing $[L_1]_{\beta,\beta}$.

(b.) Check that $([L_{e_1}]_{\beta,\beta})^2 = 0$, $([L_{e_2}]_{\beta,\beta})^2 = 0$ and $[L_{e_1}]_{\beta,\beta}[L_{e_2}]_{\beta,\beta} = [L_{e_1 \wedge e_2}]_{\beta,\beta}$ whereas $[L_{e_2}]_{\beta,\beta}[L_{e_1}]_{\beta,\beta} = -[L_{e_1 \wedge e_2}]_{\beta,\beta}$

Remark: *I know that part (b.) will work out since we can easily calculate in general that $L_x \circ L_y(z) = L_x(y \wedge z) = x \wedge y \wedge z = L_{x \wedge y}(z)$. The neat thing here is that if you forget about the abstract $e_1 \wedge e_2$ and just think about these 4×4 matrices then you can see that the algebra of the wedge product on $\mathbb{R}^2$ is easily implemented with 4×4 matrices. Generally, we can follow much the same construction to build the wedge product algebra on $\mathbb{R}^n$ by representing it on $2^n \times 2^n$ matrices. This gives us one concrete construction of the exterior algebra. To be honest, in Math 422 there is a more robust and elegant manner in which the exterior algebra can be constructed in terms of an appropriate quotient space by an ideal which implements the core identity of the wedge product. It turns out that formal algebraic object is isomorphic to the far more concrete set of completely antisymmetric multilinear maps. Both of these are known as* **tensors**, *the algebraic quotient idea is how mathematicians tend to construct such objects, the multilinear maps underlie the calculations typical of Physicists who use tensors to formulate foundational equations of nature. Our application here is less ambitious, I merely wish to show you how we can use the exterior algebra with its wedge product to deconstruct the properties of the determinant. Hopefully this also serves to give you a good introduction to associative algebras.*

Problem 107 Let $\mathcal{A} = \mathbb{R}^n$ be an associative unital algebra with multiplication $\star : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which defines a bilinear map such that

$$u \star (v \star w) = (u \star v) \star w$$

and there exists $1_\mathcal{A} \in \mathcal{A}$ for which $1_\mathcal{A} \star v = v = v \star 1_\mathcal{A}$. We define $\ell_v : \mathcal{A} \to \mathcal{A}$ by $\ell_v(x) = v \star x$ for all $x \in \mathcal{A}$.

(a.) Prove $\ell_v \circ \ell_w = \ell_{v \star w}$ for all $v, w \in \mathcal{A}$,

(b.) Show $\ell_v \in \mathcal{L}(\mathcal{A})$ for each $v \in \mathcal{A}$,

(c.) Define $\mathbf{M}(v) = [\ell_v]$ and show $\Psi : \mathcal{A} \to \mathbb{R}^{n\times n}$ defined by $\Psi(v) = \mathbf{M}(v)$ defines a linear map for which $\Psi(v \star w) = \mathbf{M}(v)\mathbf{M}(w)$

(d.) In the algebra $\mathcal{A}$ an element x is called a **unit** if there exists $x^{-1} \in \mathcal{A}$ for which $x \star x^{-1} = 1_{\mathcal{A}} = x^{-1} \star x$. Show $x \in \mathcal{A}$ is a unit if and only if $\mathbf{M}(x)$ is an invertible matrix. In particular, $x \in \mathcal{A}$ is a unit if and only if

$$det[x \star e_1 | x \star e_2 | \cdots | x \star e_n] \neq 0.$$

Problem 108 Let $\mathcal{H}_3 = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ define the 3-hyperbolic numbers. This notation indicates $e_1 = 1$, $e_2 = j$ and $e_3 = j^2$ and we use juxtaposition to denote the multiplication of $\mathcal{H}_3$ where $j^3 = 1$ by definition. We define:

$$(a + bj + cj^2)(x + jy + j^2z) = ax + bz + cy + (ay + bx + cz)j + (az + by + cx)j^2$$

(a.) Find $\mathbf{M}(a + bj + cj^2)$,

(b.) Determine the condition required for $a + bj + cj^2$ to be a unit and find the formula for $(a + bj + cj^2)^{-1}$ in that case.

Problem 109 Suppose $\mathcal{A} = \mathbb{R}^4$ is given the noncommutative product induced from the multiplication of 2×2 matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t & x \\ y & z \end{bmatrix} = \left[\begin{array}{c|c} at + by & ax + bz \\ \hline ct + dy & cx + dz \end{array}\right].$$

We define,

$$(a, b, c, d) \star (t, x, y, z) = (at + by, ax + bz, ct + dy, cx + dz).$$

(a.) Find $\mathbf{M}(a, b, c, d)$,

(b.) Determine the condition required for (a, b, c, d) to be a unit and find the formula for $(a, b, c, d)^{-1}$ in that case.

Problem 110 Let $V = \mathbb{R}^n$ and consider the exterior algebra $\Omega V = \oplus_{p=0}^{n} \Lambda^p V$ then we denote the multiplication on ΩV by $\wedge$ as we have discussed. An interesting mapping on ΩV is given by **Hodge duality** which we denote by $\star$. Given $\sigma \in \Lambda^p V$ we define $\star\sigma \in \Lambda^{n-p} V$ to be the unique $(n-p)$-vector for which:

$$\boxed{\sigma \wedge \star\sigma = \|\sigma\|^2 e_1 \wedge \cdots \wedge e_n.}$$

Here $\|\sigma\|$ denotes the length of $\sigma \in \Lambda^p V$ thought of as an $\binom{n}{k}$ vector. For example,

$$\sigma = e_1 \wedge e_2 + 3e_1 \wedge e_3 + 8e_2 \wedge e_3 \quad \Rightarrow \quad \|\sigma\| = \sqrt{1^2 + 3^2 + 8^2}.$$

or

$$\sigma = 3e_1 \wedge e_2 \wedge e_3 + 4e_2 \wedge e_4 \wedge e_5 \quad \Rightarrow \quad \|\sigma\| = \sqrt{3^2 + 4^2} = 5$$

Since $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$ has length one we can easily determine the Hodge dual of the basic p-vectors:

$$\sigma = e_1 \quad \Rightarrow e_1 \wedge \star\sigma = e_1 \wedge e_2 \wedge e_3 \quad \Rightarrow \quad \star\sigma = e_2 \wedge e_3.$$

Linearly extending, we can easily deduce for $\mathbb{R}^3$ that $\star\omega_{\vec{A}} = \Phi_{\vec{A}}$ whereas $\star\Phi_{\vec{A}} = \omega_{\vec{A}}$ and $\star 1 = e_1 \wedge e_2 \wedge e_3$. I found the boxed formula about a fairly easy to understand method to calculate Hodge duality. That said, there is a way to understand it in terms of musical morphism which is rather beautiful. Let us develop the idea, hang with me it takes some steps.

Let $\alpha, \beta \in \Lambda^p V$ then $\alpha = \sum_I \alpha_I e_I$ and $\beta = \sum_J \alpha_J e_J$ where the sum over I, J are over all strictly increasing multi-indices of length p; $I = (i_1, \dots, i_p)$ where $1 \leq i_1 < i_2 < \cdots < i_p \leq n$. We define $g_p : \Lambda^p V \times \Lambda^p V \to \mathbb{R}$ via

$$g_p(\alpha, \beta) = \sum_I \alpha_I \beta_I = \underbrace{\sum \cdots \sum}_{1 \leq i_1 < \cdots < i_p \leq n} \alpha_{i_1, \dots, i_p} \beta_{i_1, \dots, i_p}$$

Notice $\|\alpha\| = \sqrt{g_p(\alpha, \alpha)}$ and you can easily verify g forms an inner product.

Next consider $\alpha \in \Lambda^p V$ and $\gamma_1, \gamma_2 \in \Lambda^{n-p} V$. Since $\Lambda^n V = span\{e_1 \wedge \cdots \wedge e_n\}$, we observe there exists $c_\alpha(\gamma_1), c_\alpha(\gamma_2) \in \mathbb{R}$ for which

$$\alpha \wedge \gamma_1 = c_\alpha(\gamma_1) e_1 \wedge \cdots \wedge e_n \qquad \& \qquad \alpha \wedge \gamma_2 = c_\alpha(\gamma_2) e_1 \wedge \cdots \wedge e_n.$$

Consequently, for $b \in \mathbb{R}$, we can show $c_\alpha(b\gamma_1 + \gamma_2) = bc_\alpha(\gamma_1) + c_\alpha(\gamma_2)$ thus $c_\alpha \in (\Lambda^{n-p} V)^*$. Since g_{n-p} is an inner product on $\Lambda^{n-p} V$ we apply the musical morphism to select $\sharp c_\alpha \in \Lambda^{n-p} V$, the unique $(n-p)$-vector for which:

$$g(\sharp c_\alpha, \gamma) = c_\alpha(\gamma).$$

We can define the Hodge dual by $\boxed{\star \alpha = \sharp c_\alpha}$, we can prove $\star : \Lambda^p V \to \Lambda^{n-p} V$ defines an isomorphism. This is not surprising since we know that $\binom{n}{p} = \binom{n}{n-p}$ hence $dim(\Lambda^p V) = dim(\Lambda^{n-p} V)$. If we think about $\gamma = e_J$ where $|J| = n - p$ then $\alpha \wedge e_J = c_\alpha(e_J) e_1 \wedge \cdots \wedge e_n$. Then, if we set $\alpha = e_I$ where $|I| = p$ then $e_I \wedge e_J = c_{e_I}(e_J) e_1 \wedge \cdots \wedge e_n$. Hence

$$c_{e_I}(e_J) = \varepsilon(I, J) = \begin{cases} 1, & I \cap J = \emptyset, \sigma(I, J) = (1, \dots, n) \text{ and } \sigma \text{ an even permutation} \\ -1, & I \cap J = \emptyset, \sigma(I, J) = (1, \dots, n) \text{ and } \sigma \text{ an odd permutation} \\ 0, & I \cap J \neq \emptyset \end{cases}$$

However, $g(\star\alpha, e_J) = c_\alpha(e_J)$ implies $\star\alpha = \sum_J c_\alpha(e_J) e_J$. In particular,

$$\star e_I = \sum_J \varepsilon(I, J) e_J$$

which reveals, in terms of the completely antisymmetric symbol,

$$\star e_{i_1, \dots, i_p} = \epsilon_{i_1, \dots, i_p, j_1, \dots, j_{n-p}} e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}$$

where $\{i_1, \dots, i_p, j_1, \dots, j_{n-p}\} = \{1, 2, \dots, n\}$. For example,

$$\star e_1 = \epsilon_{123\dots n} e_2 \wedge e_3 \wedge \cdots \wedge e_n = e_2 \wedge e_3 \wedge \cdots \wedge e_n$$

$$\star e_2 = \epsilon_{213\dots n} e_1 \wedge e_3 \cdots \wedge e_n = -e_1 \wedge e_3 \wedge \cdots \wedge e_n$$

Or, to give a 7-dimensional example,

$$\star e_3 \wedge e_5 \wedge e_7 = \epsilon_{3571246} e_1 \wedge e_2 \wedge e_4 \wedge e_6 = -\epsilon_{1234567} e_1 \wedge e_2 \wedge e_4 \wedge e_6 = -e_1 \wedge e_2 \wedge e_4 \wedge e_6.$$

I'll circle back to our original boxed equation to see if the calculation above agrees with the original method. Let $\sigma = e_3 \wedge e_5 \wedge e_7$ then

$$\begin{aligned}
\sigma \wedge \star\sigma &= (e_3 \wedge e_5 \wedge e_7) \wedge (-e_1 \wedge e_2 \wedge e_4 \wedge e_6) \\
&= -e_1 \wedge e_2 \wedge e_4 \wedge e_6 \wedge e_3 \wedge e_5 \wedge e_7 \\
&= -e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_6 \wedge e_5 \wedge e_7 \\
&= e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7.
\end{aligned}$$

Enough introduction. Now the question:

(a.) Show $c_\alpha \in (\Lambda^{n-p} V)^*$ (see introduction for meaning of notation),

(b.) Show $\star$ is an isomorphism of vector spaces,

(c.) Calculate the Hodge dual $\star$ on $\mathbb{R}^4$.