Your PRINTED NAME indicates you read Chapter 2 of the notes: ____________________.

Same instructions as Mission 1. Notation $V(\mathbb{F})$ indicates V is a vector space over the field $\mathbb{F}$. Assume V has the natural field of scalars unless otherwise instructed, for example $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ whereas $\mathbb{C}^n$ is by default a vector space over $\mathbb{C}$.

Problem 11 If W is a subspace of $V(\mathbb{F})$ then prove it, otherwise explain why it cannot be a subspace

(a.) $V = \mathbb{R}^2$ with $W = \{(x, y) \in \mathbb{R}^2 \mid x + y = 1\}$

(b.) $V = \mathcal{F}(\mathbb{R})$ with $W = \{f(t) \in \mathcal{F}(\mathbb{R}) \mid f(3) = 0\}$

(c.) $V = \mathbb{C}^2$ with $W = \{(z, w) \in \mathbb{C}^2 \mid Re(z) + Im(w) = 0\}$

(d.) $V(\mathbb{R}) = \mathbb{C}^{m\times n}$ with $W = \{A \in \mathbb{C}^{m\times n} \mid A_{m1} = iA_{1n}\}$

(e.) $V = \mathbb{R}[x, y]$ with $W = \{c_1 + c_2xy + c_3y^2 \mid c_1, c_2, c_3 \in \mathbb{Z}_3\}$

(f.) $V = \mathbb{Q}^3$ with $W = \{(a + b, b + c, c + a) \mid a, b, c \in \mathbb{Q}\}$

Problem 12 Let M, N, P be subspaces of V over the field $\mathbb{F}$.

(a.) Suppose M is contained in P. Prove $P \cap (M + N) = M + (P \cap N)$

(b.) Show $P \cap (M + N) \neq (P \cap M) + (P \cap N)$ if we are not given that $M \subseteq P$.

Problem 13 Let V_1 and V_2 be vector spaces over $\mathbb{F}$. Define $V = V_1 \times V_2$ and let vector addition and scalar multiplication be defined as follows:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \qquad \& \qquad c(x_1, x_2) = (cx_1, cx_2).$$

Prove V is a vector space over $\mathbb{F}$.

Problem 14 Let V be a vector space over $\mathbb{R}$. We define the **complexification** of V as $V_{\mathbb{C}} = V \times V$ where we define $+ : V_{\mathbb{C}} \times V_{\mathbb{C}} \to V_{\mathbb{C}}$ and scalar multiplication $\mathbb{C} \times V_{\mathbb{C}} \to V_{\mathbb{C}}$ via

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \qquad \& \qquad (a + ib)(x_1, x_2) = (ax_1 - bx_2, ax_2 + bx_1).$$

Prove $V_{\mathbb{C}}$ is a complex vector space.
Note: we often use the notation $(x_1, x_2) = x_1 + ix_2$ in which case the scalar multiplication rule above is given by $(a + ib)(x_1 + ix_2) = ax_1 - bx_2 + i(ax_2 + bx_1)$. Notice this can be remembered by the simple rule $i^2 = -1$ and procede by the usual rules of algebra.

Problem 15 Suppose W is a vector space over $\mathbb{F}$ and S is a set. We define $\mathcal{F}(S, W)$ to be the set of all functions from S to W. Show $\mathcal{F}(S, W)$ is a vector space over $\mathbb{F}$[1]. Also, if you're given vector spaces V and W both over $\mathbb{F}$, then show $\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ linear}\}$ is a subspace of $\mathcal{F}(V, W)$.

[1] define how we add vectors and scalar multiply vectors in $\mathcal{F}(S, W)$ and show why several of the axioms hold true, you don't have to do all the axioms

Problem 16 Let M, N be subspaces of $V(\mathbb{F})$ for which $M + N = V$ and $M \cap N = \{0\}$ then we say V is the **direct sum** of M and N and write $V = M \oplus N$. Prove the following are equivalent:

(i.) $V = M \oplus N$,

(ii.) for each $x \in V$, there exist unique elements $y \in M$ and $z \in N$ such that $x = y + z$.

Problem 17 We denote $\mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{R}, \mathbb{R})$. Let $T \subseteq \mathbb{R}$.

(a.) Prove $V = \{f \in \mathcal{F}(\mathbb{R}) \mid f(t) = 0 \text{ for all } t \in T\}$ is a real vector space.

(b.) Observe $M = \{f \in V \mid f(x) = 0 \text{ for } x < 0\}$ and $N = \{f \in V \mid f(x) = 0 \text{ for } x \geq 0\}$ are subspaces of V. Prove $V = M \oplus N$.

Problem 18 Given $S = \{u, v, w\}$ is a spanning set for V, show $T = \{u, u + v, u + v + w\}$ is also a spanning set for V.

Problem 19 Given $S = \{u, v, w\}$ is a LI subset of V, show $T = \{u + 2v + w, u - 3v + w, u + v - 7w\}$ is also a LI subset of V.

Problem 20 For any $n \in \mathbb{N}$, show $S = \{Id, D, D^2, \ldots, D^n\}$ is a linearly independent subset of $V = \mathcal{L}(\mathbb{R}[x], \mathbb{R}[x])$. Here we denote $D[f(x)] = f'(x)$ and $D^2[f(x)] = f''(x)$ whereas $Id(f(x)) = f(x)$ for $f(x) \in \mathbb{R}[x]$.