Your PRINTED NAME indicates you read Chapter 3 of the notes: ____________________.

Same instructions as Mission 1. Notation $V(\mathbb{F})$ indicates V is a vector space over the field $\mathbb{F}$. Assume V has the natural field of scalars unless otherwise instructed, for example $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ whereas $\mathbb{C}^n$ is by default a vector space over $\mathbb{C}$.

Problem 21 Let $A \subseteq V$ where V is a vector space over the field $\mathbb{F}$. We define A to be Linearly Independent (LI) if every finite collection $\{a_1, \dots, a_k\} \subseteq A$ has the property that $c_1a_1 + \cdots + c_ka_k = 0$ for $c_1, \dots, c_k \in \mathbb{F}$ implies $c_j = 0$ for $j = 1, \dots, k$. Prove the following elementary lemmas:

(a.) A is LI if and only if for every $x \in A$, $x \notin span(A - \{x\})$,

(b.) $A = \emptyset$ is LI,

(c.) If A is LI and $B \subseteq A$ then B is LI,

(d.) If $T : V \to W$ is an injective linear map and $A \subseteq V$ is LI, then $T(A)$ is LI in W.

Problem 22 Let $B = E_{11} - E_{22}$ be a 2×2 matrix and define $W = \{A \in \mathbb{F}^{2\times 2} \mid AB = 0\}$. Prove $W \leq \mathbb{F}^{2\times 2}$ and

(a.) if $\mathbb{F} = \mathbb{R}$, find a basis for W and calculate $dim(W)$. What is the cardinality of W ?

(b.) if $\mathbb{F} = \mathbb{Z}_3$, find a basis for W and calculate $dim(W)$. What is the cardinality of W ?

Problem 23 Define subspaces of $P_3(\mathbb{R})$ as follows:

$$U = \{f(x) \in P_3(\mathbb{R}) \mid f(2) = 0\} \qquad \& \qquad V = \text{span}\{3 + x, 3 - x\}.$$

Find a basis for $U \cap V$. Use a theorem to decide what $U + V$ must be on the basis of dimensional arguments.

Problem 24 Let $W = span\{1 + x, x + x^3\}$. Find all possible bases for $P_3(\mathbb{R})$ which are formed by extending $\{1 + x, x + x^3\}$.

Problem 25 Let $V = C^0(\mathbb{R})$ be the vector space of all continuous functions on $\mathbb{R}$. Define

$$W = \left\{ f \in W \mid \lim_{x\to 1} f(x) = 0 \right\}.$$

Show $W \leq V$ and prove that W is infinite dimensional.

Problem 26 Let $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$ where V_1 and V_2 are vector spaces over $\mathbb{F}$. Prove:

(a.) If S_1 and S_2 are LI then $(S_1 \times \{0\}) \cup (\{0\} \times S_2)$ is LI subset of $V_1 \times V_2$.

(b.) If $V_1 = \text{span}(S_1)$ and $V_2 = \text{span}(S_2)$ then $\text{span}\left((S_1 \times \{0\}) \cup (\{0\} \times S_2)\right) = V_1 \times V_2$.

(c.) If V is a real vector space of finite dimension then $dim_{\mathbb{R}}(V_{\mathbb{C}}(\mathbb{R})) = 2dim_{\mathbb{R}}(V)$.

Problem 27 Let $W = \{A \in \mathbb{R}^{n\times n} \mid A^T = A \;\&\; tr(A) = 0\}$. Prove $W \leq \mathbb{R}^{n\times n}$ and calculate $dim(W)$.

Problem 28 Suppose $T : P_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ is defined by $T(1) = E_{11} + E_{22}$ and $T(x) = 2E_{11} + 2E_{22}$ and $T(x^2) = E_{12} - E_{21}$ extended linearly.

(a.) If $\beta = \{1, x, x^2\}$ and $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ then calculate $[T]_{\beta,\gamma}$

(b.) Find a basis for $Ker(T)$

(c.) Find a basis for $Im(T) = T(P_2(\mathbb{R}))$

(d.) Find a subspace W for which $T|_W : W \to Im(T)$ is an isomorphism

(e.) Verify the rank nullity theorem both for T and for $T|_W$

Problem 29 Consider $V = \mathbb{R}[x]$. Find linear transformations on V such that

(a.) $T : V \to V$ is injective, but not surjective

(b.) $T : V \to V$ is surjective, but not injective.

Problem 30 Let U, V, W be vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove:

(a.) If $S \circ T$ is surjective then S is surjective,

(b.) If $S \circ T$ is injective then T is injective.