Your PRINTED NAME indicates you read Chapter 3 of the notes: ____________________.

Same instructions as Mission 1. Notation $V(\mathbb{F})$ indicates V is a vector space over the field $\mathbb{F}$. Assume V has the natural field of scalars unless otherwise instructed, for example $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ whereas $\mathbb{C}^n$ is by default a vector space over $\mathbb{C}$.

Problem 31 Let $V = \mathbb{C} \times P_1(\mathbb{R})$ be the real vector space formed by pairs of the form $(a + ib, cx + d)$.

(a.) Find a basis β for V as a real vector space.

(b.) If possible, find an isomorphism from V to $\mathbb{R}^{n\times n}$ for appropriate n.

Problem 32 Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which $T(v_1) = 3v_1$ and $T(v_2) = 2v_2$ and $T(v_3) = v_3$ for nonzero vectors v_1, v_2, v_3. Prove:

(a.) $\{v_1, v_2, v_3\}$ is linearly independent.

(b.) If $\beta = \{v_1, v_2, v_3\}$ then find $[T]_{\beta,\beta}$ and find $[T]$ in terms of the given data.

(c.) Calculate $\mathrm{tr}([T])$ and $\det([T])$.

Problem 33 Suppose $T : P_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ is defined by $T(1) = E_{11} + E_{22}$ and $T(x) = 2E_{11} + 2E_{22}$ and $T(x^2) = E_{12} + E_{21}$ extended linearly.

(a.) Find the formula for $T(ax^2 + bx + c)$.

(b.) If $\beta = \{x^2, x, 1\}$ and $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ then find $[T]_{\beta,\gamma}$.

(c.) Find a basis for $Ker(T)$.

(d.) Construct bases δ for $P_2(\mathbb{R})$ and σ for $\mathbb{R}^{2\times 2}$ for which $[T]_{\delta,\sigma} = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$ where k is the rank of T.

Problem 34 Let $W_1 = \{tI \mid t \in \mathbb{R}\}$ where I is the $n \times n$ identity matrix and W_2 is defined by $W_2 = \{A \in \mathbb{R}^{n\times n} \mid A^T = A,\ tr(A) = 0\}$ and $W_3 = \{A \in \mathbb{R}^{n\times n} \mid A^T = -A\}$.

(a.) Show $W_1 + W_2 + W_3 = \mathbb{R}^{n\times n}$.

(b.) Show W_1, W_2, W_3 are independent subspaces.

Problem 35 Let $T(f(x)) = f(x) + xf'(x)$ for $f(x) \in P_3(\mathbb{R})$. Let $\beta_1 = \{1, x^2\}$ and $\beta_2 = \{x, x^3\}$ provide bases for $W_1 = \mathrm{span}(\beta_1)$ and $W_2 = \mathrm{span}(\beta_2)$.

(a.) show W_1 and W_2 are invariant subspaces of T,

(b.) Find $[T_{W_1}]_{\beta_1,\beta_1}$ and $[T_{W_2}]_{\beta_2,\beta_2}$.

(c.) verify $[T]_{\beta,\beta} = [T_{W_1}]_{\beta_1,\beta_1} \oplus [T_{W_2}]_{\beta_2,\beta_2}$ where $\beta = \beta_1 \cup \beta_2$.

Problem 36 Let $A \in \mathbb{F}^{n\times n}$ and define $T : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ by $T(A) = A + A^T$

(a.) show that $S_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$ is an invariant subspace of T.

(b.) show that $A_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$ is contained in $\mathrm{Ker}(T)$.

(c.) Let β_s and β_a be bases for the symmetric and antisymmetric $n \times n$ matrices over $\mathbb{F}$. Form basis $\beta = \beta_s \cup \beta_a$ and find the block-structure of the matrix $[T]_{\beta,\beta}$.

Problem 37 For each W and V given below, find a basis for V/W:

(a.) $W = \{f(x) \in P_3(\mathbb{R}) \mid f(3) = 0, f'(3) = 0\}$ and $V = P_3(\mathbb{R})$,

(b.) $W = \{A \in \mathbb{R}^{2\times 2} \mid A^T = -A\}$ and $V = \mathbb{R}^{2\times 2}$,

(c.) $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ and $V = \mathbb{R}^3$.

Problem 38 Let S be a LI subset of V such that $S \cap W = \emptyset$. Show $\{s + W \mid s \in S\}$ forms a LI subset of V/W.

Problem 39 Let $T : V \to V$ be a linear mapping such that $T^2 = T$, and let $M = Im(T)$ and $N = Ker(T)$. Prove that $M = \{x \in V \mid T(x) = x\} = Ker(T - Id)$ and that $V = M \oplus N$.

Problem 40 Let U, V, W be vector spaces over $\mathbb{F}$ and suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. Prove $Ker(ST) = T^{-1}(Ker(S))$.