Your PRINTED NAME indicates you read Chapter 3 of the notes: ____________________.

Same instructions as Mission 1. Notation $V(\mathbb{F})$ indicates V is a vector space over the field $\mathbb{F}$. Assume V has the natural field of scalars unless otherwise instructed, for example $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ whereas $\mathbb{C}^n$ is by default a vector space over $\mathbb{C}$.

Problem 41 Provide an explicit isomorphism $\Psi : V \to W$ for the following:

(a.) $V = \mathbb{F} \times \mathbb{F}^{m\times n}$ and $W = P_k(\mathbb{F})$ where $\mathbb{F}$ is a field,

(b.) $V(\mathbb{R}) = \mathbb{C}^{n^2}$ and $W = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$,

(c.) $V = \{A \in \mathbb{R}^{2\times 2} \mid A^T = -A\}$ and $W = \mathbb{R}[x]/M$ where $M = span\{x^2, x^3, x^4, \dots\}$,

(d.) $V = (\,\mathbb{R}^{2\times 2}\,)[x]$ and $W = (\,\mathbb{R}[x]\,)^{2\times 2}$,

(e.) $V = \mathbb{R}^{m\times n}$ and $W = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

Problem 42 Suppose $V = M \oplus N$. Prove $V/M \times V/N \cong M \times N$.

Problem 43 For U, V, W finite dimensional vector spaces over the field $\mathbb{F}$. Argue $\mathcal{L}(U,W) \times \mathcal{L}(V,W)$ is isomorphic to $\mathcal{L}(U \times V, W)$ via dimensional arguments. Also, construct a mapping $\Psi : \mathcal{L}(U,W) \times \mathcal{L}(V,W) \to \mathcal{L}(U \times V, W)$ and prove your map is an explicit isomorphism.

Problem 44 Let $V = \mathcal{F}(\mathbb{R})$ be the set of all functions on $\mathbb{R}$. Let W_1 be the set of all even functions $(f(x) = f(-x))$ and let W_2 be the set of odd functions $(f(-x) = -f(x))$. Show $V/W_1 \cong W_2$.

Problem 45 Define $D^k : \mathbb{R}[x] \to \mathbb{R}[x]$ by $D[f(x)] = d^k f/dx^k$. Show D^k is a surjection and find the formula for the inverse of the mapping $\overline{D^k} : \mathbb{R}[x]/Ker(D^k) \to \mathbb{R}[x]$ where $\overline{D^k}(f(x) + Ker(D^k)) = D^k(f(x))$ for each $f(x) \in \mathbb{R}[x]$.

Problem 46 Let $V = span_{\mathbb{F}}(\beta)$ where $\beta = \{v_1, \dots, v_n\}$ is a basis. Define $\beta^* = \{v^1, \dots, v^n\}$, the **dual basis** by linearly extending the rules $v^i(v_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$ in order that $v^i : V \to \mathbb{F}$ is a linear map for each $i = 1, \dots, n$.

(a.) prove β^* is LI,

(b.) show $\alpha \in V^*$ implies $\alpha = \displaystyle\sum_{i=1}^{n} \alpha(v_i)v^i$ hence $span(\beta^*) = V^*$

(c.) prove $\Phi_\beta = (v^1, \dots, v^n)$, that is the dual basis form component functions of the β coordinate map

Problem 47 Since $tr : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a linear map it follows $tr \in (\mathbb{R}^{n\times n})^*$. Express the trace in terms of the dual-basis of the standard basis $\{E_{ij}\}$ which we will denote $\{E^{ij}\}$ where $E^{kl}(E_{ij}) = \delta_{ik}\delta_{jl}$ serves to define the dual basis.

Problem 48 Suppose W is a subspace of $V(\mathbb{F})$. Let $ann(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \text{ for all } w \in W\}$, this is the **annihilator** of W in the dual space.

(a.) Prove $ann(W) \leq V^*$

(b.) If $W_1 \leq W_2 \leq V$ then prove $ann(W_2) \leq ann(W_1) \leq V^*$

(c.) If V is finite dimensional then $dim(ann(W)) = dim(V) - dim(W)$.

Problem 49 Suppose V, W are finite dimensional vector spaces over $\mathbb{F}$ and let $G : V \to W$ be a linear mapping. Define the **dual** of G (this has various names in various places) by

$$(G^*(\alpha))(x) = \alpha(G(x))$$

for each $x \in V$ and $\alpha \in W^*$. Prove $G^* \in \mathcal{L}(W^*, V^*)$. Determine how the injectivity or surjectivity of G relate to the injectivity or surjectivity of G^*. Also, how does the rank of G and G^* relate ? Prove your assertions.

Problem 50 Suppose V is a finite dimensional vector space and $V = W_1 \oplus W_2$. Does it follow that $ann(W_1) \oplus ann(W_2) = V^*$? Prove or disprove.