Your PRINTED NAME indicates you read Chapter 4 of the notes: ____________________.

Problem 51 Suppose $V = V_1 \oplus \cdots \oplus V_s$ where $V_j \neq 0$ for $j = 1, \dots, s$. Show there exist $E_1, \dots, E_s \in \mathcal{L}(V)$ such that $E_1 + \cdots + E_s = Id_V$ and $E_i E_j = 0$ for $i \neq j$ and $V_j = E_j(V)$ for $j = 1, \dots, s$.

Problem 52 Determine if A is diagonalizable over $\mathbb{F}$ and calculate A^{-1} as a polynomial in A if possible.

(a.) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ with $\mathbb{F} = \mathbb{C}$,

(b.) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ with $\mathbb{F} = \mathbb{Q}$,

(c.) Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ with $\mathbb{F} = \mathbb{R}$.

(d.) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ with $\mathbb{F} = \mathbb{C}$.

Problem 53 Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$. Let $T = L_A$ and find the standard matrices of transformations E_1, E_2, E_3 for which $E_j(\mathbb{R}^3) = \text{Ker}(T - \alpha_j)$ for $j = 1, 2, 3$ and

$$\mathbb{R}^3 = E_1(\mathbb{R}^3) \oplus E_2(\mathbb{R}^3) \oplus E_3(\mathbb{R}^3).$$

where E_1, E_2, E_3 are idempotent and pairwise commuting with $E_1E_2 = E_1E_3 = E_2E_3 = 0$.

Problem 54 Let V be an n-dimensional vector space over the field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$. Let $P_T(x) = det(x - T)$ define the characteristic polynomial of T.

(a.) Prove there are at most n-distinct zeros of P_T

(b.) If $\lambda \in \mathbb{F}$ and $P_T(\lambda) = 0$ then show $dim(\mathcal{E}_\lambda) \geq 1$. Here we define the eigenspace of eigenvalue λ as $\mathcal{E}_\lambda = Ker(T - \lambda)$.
Importance: notice, this proof allows us to identify eigenvalues with zeros of the characteristic polynomial since each zero begets an eigenvector.

(c.) Prove T is diagonalizable if and only if P_T has n-zeros (allowing repeats).

Problem 55 Let $V \neq 0$ be a complex vector space of finite dimension and let $T : V \to V$ be a linear map with $T^2 = 1$. Define $V_+ = \{x \in V \mid T(x) = x\}$ and $V_- = \{x \in V \mid T(x) = -x\}$. Prove $V = V_+ \oplus V_-$.

Problem 56 Let V be an n-dimensional vector space over the field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$. Let $P_T(x) = det(x - T)$. Suppose $\lambda_1, \dots, \lambda_k$ are distinct zeros of $P_T(x)$ with corresponding eigenspaces $\mathcal{E}_{\lambda_j} = Ker(T - \lambda_j)$ for $1 \leq j \leq k$.

(a.) Let $W = \mathcal{E}_{\lambda_1} + \cdots + \mathcal{E}_{\lambda_k}$. Show W is an invariant subspace of T. Also, define $T_W : W \to W$ by $T_W(x) = T(x)$ for each $x \in W$. Show $det(x - T_W)$ divides $P_T(x)$.

(b.) Show $W = \mathcal{E}_{\lambda_1} \oplus \cdots \oplus \mathcal{E}_{\lambda_k}$.

(c.) Prove T_W is diagonalizable.

Problem 57 Let V be an n-dimensional vector space over the field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$. Let $P_T(x) = det(x - T)$. Suppose $\lambda_1, \dots, \lambda_k$ are distinct zeros of $P_T(x)$ with corresponding eigenspaces $\mathcal{E}_{\lambda_j} = Ker(T - \lambda_j)$ for $1 \leq j \leq k$. Also, define $W = \mathcal{E}_{\lambda_1} + \cdots + \mathcal{E}_{\lambda_k}$ as was studied in the previous problem. We suppose $dim(W) < dim(V)$ hence there exists basis $\beta_W = \{w_1, \dots, w_m\}$ for W where $m < n$. Furthermore, let $\beta = \{w_1, \dots, w_m, v_1, \dots, v_{n-m}\}$ be a basis for V.

(a.) Let $M = span\{v_1, \dots, v_{n-m}\}$ and show $W \oplus M = V$.

(b.) Define $\overline{T} : V/M \to V/M$ by $\overline{T}(x + M) = T(x) + M$. Show $\overline{T}$ is well-defined.

(c.) Prove $\overline{T}$ is diagonalizable.

Problem 58 Suppose V is a finite dimensional real vector space and $T : V \to V$ is a linear transformation. Determine if the statements below are True, False, or Possible. Explain your choice.

(a.) If $T^2 = T$ then T is diagonalizable.

(b.) If $T^2 + 1 = 0$ then T is diagonalizable.

(c.) If $T^2 + 2T + 1 = 0$ then T is diagonalizable.

Note: I'm using the notation 1 *as an abbreviation for* Id_V*, this should be clear from the context.*

Problem 59 Let V be a finite dimensional complex vector space and suppose $T : V \to V$ is a linear transformation with Jordan decomposition $T = D + N$ where D is diagonalizable and N is nilpotent. We're also given that N and D commute and there exist polynomials $f(t), g(t) \in \mathbb{C}[t]$ for which $f(T) = D$ and $g(T) = N$.

(a.) Suppose $A, B \in \mathcal{L}(V)$ are commuting linear transformations. Let $h(t) \in \mathbb{C}[t]$, prove $h(A)B = Bh(A)$.

(b.) Prove $TX = XT$ if and only if $DX = XD$ and $NX = XN$.

Problem 60 Consider $V = \text{span}_{\mathbb{R}}\{\cosh(x), \sinh(x), \cos(x), \sin(x)\}$ and let $D = d/dx$. Find the eigenvalues of T ($T_{\mathbb{C}}$ in **(c.)**) and decompose V ($V_{\mathbb{C}}$ in **(c.)**) as the direct sum of eigenspaces.

(a.) $T = D^2 + 1$.

(b.) $T = D(D^2 + 1)$.

(c.) $T_{\mathbb{C}} = D_{\mathbb{C}}$ where $D_{\mathbb{C}}(f + ig) = D[f] + iD[g]$ for all $f + ig \in V_{\mathbb{C}}$.

Hint: it is helpful to note $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ *and* $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ *whereas* $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ *and* $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.