Your PRINTED NAME indicates you read Chapter 4 of the notes: ____________________.

Problem 61 Find the Jordan form of the given matrices and determine which matrices are similar:

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 62 Suppose A is a 6×6 real matrix with characteristic polynomial $p(t) = (t-3)^3(t-2)^2(t-1)$. What are the possible Jordan forms associated to A. For each form determine the minimal polynomial for A.

Problem 63 Let $T : V \to V$ have basis $\beta = \{v_1, \dots, v_n\}$ for which the matrix of T is in Jordan form:

$$[T]_{\beta,\beta} = J_4(3) \oplus J_2(3) \oplus J_1(3) \oplus J_1(3) \oplus J_4(6)$$

Select vectors from β to construct the basis for each eigenspace and generalized eigenspace for T. That is, find $\beta_j \subset \beta$ for which $\mathcal{E}_{\lambda_j} = \text{span}(\beta_j)$ and $\gamma_j \subset \beta$ for which $\text{span}(\gamma_j) = K_\lambda$ for each eigenvalue of T.

Problem 64 Suppose $A \in \mathbb{R}^{14\times 14}$ and $rk((A-7I)^5) = 4$, $rk((A-7I)^4) = 4$, $rk((A-7I)^3) = 5$, $rk((A-7I)^2) = 7$, $rk(A-7I) = 10$ and $rk((A-3I)^2) = 10$, $rk(A-3I) = 12$. Here $rk(M) = dim(Col(M)$ is the **rank** of the matrix M.

(a.) Find the Jordan form of A.

(b.) If $P_A(x) = det(xI - A)$ then find the formula for $P_A(x)$ in factored form.

(c.) Let $m_A(x)$ be the **minimal polynomial** of A; that is, let $m_A(x)$ be the monic polynomial of least degree for which $m_A(A) = 0$. Find $m_A(x)$.

Problem 65 Recall $e^{(\alpha+i\beta)t} = e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t$. Suppose T is a linear transformation on $V = \text{span}_{\mathbb{R}}\Upsilon$ where $\Upsilon = \{e^t\cos 2t, e^t\sin 2t, \cos 3t, \sin 3t\}$. Given that the complexification of T has

$$T_{\mathbb{C}}(e^{(1+2i)t}) = (4+5i)e^{(1+2i)t} \qquad \& \qquad T_{\mathbb{C}}(e^{3it}) = (7+8i)e^{3it}$$

Find the real Jordan form of T.

Problem 66 Let $A = \begin{bmatrix} -5 & 4 & 0 & -4 \\ -11 & 8 & -2 & -7 \\ 2 & -1 & 1 & 3 \\ -1 & 1 & -2 & 0 \end{bmatrix}$. You can check this matrix has eigenvalues of $\lambda = 1 \pm 2i$ repeated. In fact (and please, understand, I do **not** want you to actually find these vectors) there exist nonzero vectors $v_1 = a_1 + ib_1$ and $v_2 = a_2 + ib_2$ such that:

$$(A - (1+2i)I)v_1 = 0 \qquad \& \qquad (A - (1+2i)I)v_2 = v_1$$

If $L(x) = Ax$ then find $[L]_{\beta,\beta}$ and $[L]_{\gamma,\gamma}$ with respect to the bases $\beta = \{a_1, b_1, a_2, b_2\}$ and $\gamma = \{v_1, v_2, \overline{v_1}, \overline{v_2}\}$. Where helpful, use $\otimes$ to express your answer.

Problem 67 Suppose $T : V \to V$ has characteristic polynomial $p(x) = (x^2 + 9)^2 (x - 3)^3$. Make a table which lists the possible real Jordan forms for T. For each case, determine the minimal polynomial. For which case(s) is $T_{\mathbb{C}}$ diagonalizable? For which case(s) is T diagonalizable?

Problem 68 Let the linear map $T : V \to V$ have Jordan basis β for the finite dimensional vector space V over the field $\mathbb{F}$. Let $S = T + cId_V$ where $c \in \mathbb{F}$. Find the Jordan form of S.

Problem 69 Let $T, U \in \mathcal{L}(V)$ be diagonalizable linear maps on the finite dimensional vector space V over $\mathbb{F}$. Prove T and U are simultaneously diagonalizable iff $TU = UT$. Note, **simultaneous diagonalizability** of U and T means there exists basis β for which $[T]_{\beta,\beta}$ and $[U]_{\beta,\beta}$ are both diagonal.

Problem 70 Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $v \in V$ be a nonzero vector. Let W be the T-cyclic subspace of V generated by v:

$$W = \langle\langle v \rangle\rangle = span\{v, T(v), T^2(v), T^3(v), \dots\}.$$

(a.) W is T-invariant

(b.) Any T-invariant subspace of V containing v also contains W.

(c.) There exists $k \in \mathbb{N}$ for which $\{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$ is a basis for W.

(d.) If $dim(W) = k$ and $a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$, then the characteristic polynomial for $T_W : W \to W$ is given by

$$f(t) = (-1)^k \left(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k \right).$$