Your PRINTED NAME indicates you read Chapter 5 of the notes: ____________________.

Problem 71 Let $A \in \mathbb{R}^{n\times n}$, prove $det(e^A) = exp(tr(A))$.

Problem 72 Let $B \in \mathbb{R}^{n\times n}$ be an antisymmetric matrix: $B^T = -B$. Define $F(x) = e^B x$ for each $x \in \mathbb{R}^n$. Show that F is an orthogonal transformation.

Problem 73 Let $\text{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I\}$. Show that: If $R \in \text{SO}(3)$ and $R \neq I$ then R has only two e-vectors of unit length for which $\lambda = 1$.

Problem 74 Explain why the following formulas do **not** define an inner product:

(a.) $\langle (a,b),(c,d)\rangle = ac - bd$ on $\mathbb{R}^2$

(b.) $\langle A, B\rangle = tr(A+B)$ on $\mathbb{R}^{2\times 2}$

(c.) $\langle (z_1,z_2),(w_1,w_2)\rangle = z_1w_1 + z_2w_2$ on $\mathbb{C}^2(\mathbb{C})$

(d.) $\langle (z_1,z_2),(w_1,w_2)\rangle = z_1w_1 + z_2w_2$ on $\mathbb{C}^2(\mathbb{R})$

Problem 75 Show that $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$ defines an inner product on $\mathbb{R}[x]$. Explain why the same formula fails to define an inner product on the space of continuous real-valued functions on $\mathbb{R}$.

Problem 76 Let $\beta = \{E_{ii} \mid 1 \leq i \leq n\} \cup \{E_{ij} + E_{ji} \mid 1 \leq i < j \leq n\}$ form a basis for the symmetric matrices $S_n \leq \mathbb{R}^{n\times n}$. Show that β is an orthogonal basis with respect to the Frobenius inner product.

Problem 77 Consider $S = \{1, x, e^x\}$. Find an orthonormal basis for $W = \text{span}(S)$ where the inner product is given by $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx$

Problem 78 Let $(V, \langle , \rangle)$ be an inner product space. Prove or disprove the following assertions:

(a.) If S is a subset of orthogonal vectors in V then S is LI,

(b.) If S is a subset of orthonormal vectors in V then S is LI.

Problem 79 **Cauchy Schwarz Inequality** Let V be an inner product space over $\mathbb{F}$ (either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) with inner product $\langle , \rangle$ and induced norm $\|x\| = \sqrt{\langle x, x\rangle}$. Prove $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ for all $x, y \in V$. Let me give you a path:

(a.) show $0 \leq \|x - cy\|^2 = \|x\|^2 - c\langle y, x\rangle - \overline{c}\langle x, y\rangle + |c|^2\|y\|^2$ for all $x, y \in V$ and $c \in \mathbb{F}$,

(b.) when $y \neq 0$ can set $c = \frac{\langle x,y\rangle}{\langle y,y\rangle}$ in the inequality in $(a.)$

(c.) rearrange the inequality to derive the Cauchy Schwarz inequality in the case $y \neq 0$.

This is just a sketch, you need to connect these thoughts with a proper proof narrative.

Problem 80 **Triangle Inequality:** Let $(V, \langle , \rangle)$ be an inner product space with the usual induced norm $\|x\| = \sqrt{\langle x, x\rangle}$. Prove $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.