Your PRINTED NAME indicates you read Chapter 5 of the notes: ____________________.

Problem 81 Suppose $\{v_1, \dots, v_k\}$ is an orthogonal subset of the inner product space $(V, \langle \ , \ \rangle)$ and suppose $a_1, \dots, a_k$ are scalars. Show $\| \sum_{i=1}^k a_i v_i \|^2 = \sum_{i=1}^k |a_i|^2 \|v_i\|^2$.

Problem 82 Let W_1 and W_2 be subspaces of finite dimensional vector space V. Prove the following:

(a.) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

(b.) $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

Problem 83 Let $W = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ for a subspace in $\mathbb{R}^{2\times 2}$ with the standard Frobenius inner product $\langle A, B\rangle = \text{trace}(AB^T)$.

(a.) Find an orthonormal basis for $W^\perp$

(b.) Calculate $Proj_{W^\perp} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(c.) Calculate $Proj_W \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(d.) Find the matrix in W which is closest to $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

Problem 84 Let $S = \{x + x^2\} \subset P_2(\mathbb{R})$ where $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\, dx$. Find a basis for $S^\perp$.

Problem 85 Let $S = \{e_1 + e_2, e_{n-1} + e_n\} \subseteq \mathbb{R}^n$ where $n \geq 3$. Find an orthonormal basis for $S^\perp$. Further, given $x \in \mathbb{R}^n$, find the closest point to x in $S^\perp$.

Problem 86 Let V be the vector space of continuous real-valued functions on $[-1, 1]$. Let W_e be the subspace of even functions in V and let W_o be the subspace of odd functions in V. Prove $W_e^\perp = W_o$ with respect to $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)\, dt$.

Problem 87 Calculate T^*, it is the unique linear tranformation on an inner product space V such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$. For the given V and T find T^*:

(a.) $T(x, y) = (x + 2y, 3y)$ for $V = \mathbb{R}^2$ with the dot-product,

(b.) $T(z_1, z_2) = (3z_1 + iz_2, z_1 + (2 - 7i)z_2)$ for $V = \mathbb{C}^2$ with $\langle z, w\rangle = z^T \overline{w}$

(c.) $T(f) = f' + 3f$ where $f \in P_1(\mathbb{R})$ and $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)\, dt$

Problem 88 Formally, we have the identity: for $-1 \leq x \leq 1$

$$\boxed{x^2 = \sum_{n=0}^{\infty} \left(\frac{\langle x^2, \cos(n\pi x)\rangle}{\langle \cos(n\pi x), \cos(n\pi x)\rangle} \right) \cos(n\pi x) + \sum_{n=1}^{\infty} \left(\frac{\langle x^2, \sin(n\pi x)\rangle}{\langle \sin(n\pi x), \sin(n\pi x)\rangle} \right) \sin(n\pi x)} \quad (\star)$$

where $\langle f(x), g(x)\rangle = \int_{-1}^1 f(x)g(x)\, dx$ in this context. I say *formally* since we've left the world of finite linear algebra here. If we truncated these sums at finite n then we'd have trigonmetric approximations of x^2 within a $2n+1$ dimensional subspace of function space. However, we consider the full infinite sum and technically we should justify that the trigonmetric series converges, and, that its limit function does reproduce x^2 on $[-1,1]$. We leave the formalities to the analysis course. Show (formally) that $(\star)$ yields:

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x)$$

on $-1 \leq x \leq 1$. You can use a CAS to do the integrals which are needed. Use this result to show $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$ (the $p = 2$ series converges to this value).

Problem 89 Let V be a complex inner product space and suppose $T : V \to V$ is a skew-hermitian map in the sense $T^\dagger = -T$. We define $T^\dagger$ to be the endomorphism implicitly given by the condition $\langle T(x), y\rangle = \langle x, T^\dagger(y)\rangle$ for all $x, y \in V$. Given this data about T, prove the following:

(a.) if T has eigenvalue λ then $\lambda = i\alpha$ for some $\alpha \in \mathbb{R}$ (that is to say, the eigenvalues of T are pure-imaginary)

(b.) if $W_i = \text{Ker}(T - \lambda_i)$ and $W_j = \text{Ker}(T - \lambda_j)$ where $\lambda_i \neq \lambda_j$ are distinct e-values of T then $W_i \perp W_j$.

Problem 90 Let $T : V \to V$ be a linear transformation on the finite dimensional inner product space $(V, \langle\ ,\ \rangle)$. Show the following are equivalent:

(a.) $T^*T = Id$,

(b.) $TT^* = Id$,

(c.) $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$

(d.) If β is an orthnormal basis for V then $T(\beta)$ is an orthonormal basis for V

(e.) $\|T(x)\| = \|x\|$ for all $x \in V$