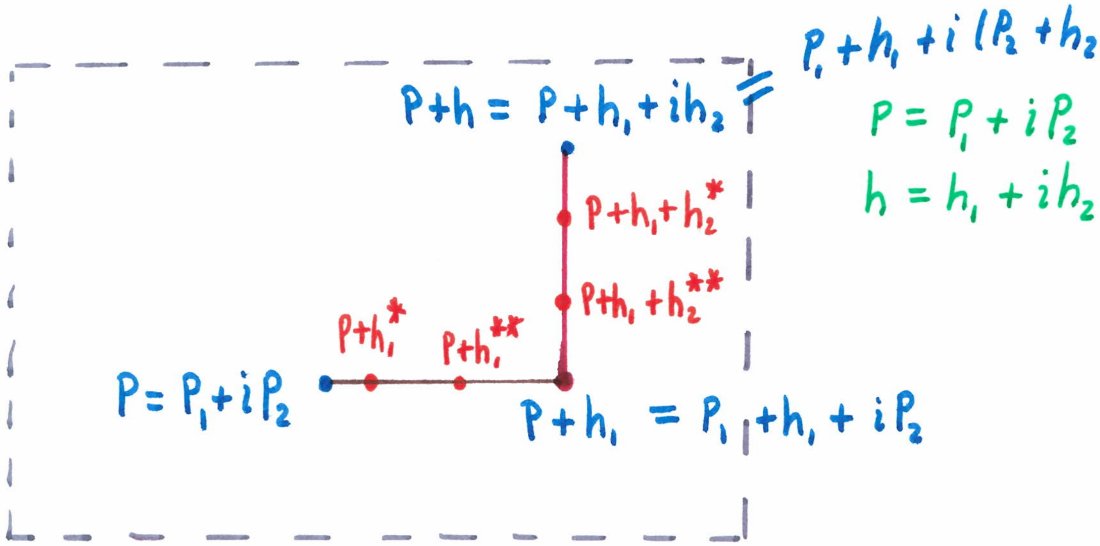
# CONTINUOUSLY DIFFERENTIABLE ⟹ DIFFERENTIABLE

We consider $f = u + iv : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $U$ is an open subset of $\mathbb{R}^2$ on which $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous and $p \in U$. For specificity's sake we take $U = (a,b) \times (c,d) \leftarrow$ rectangle




We wish to control $f(p+h) - f(p)$, notice we can express the overall $\Delta f$ in terms of the sum of a horizontal & vertical displacements.

$$\Delta f = f(p+h) - f(p) = f(p+h) - f(p+h_1) + f(p+h_1) - f(p)$$

$$= u(p+h) - u(p_1+h_1) + u(p+h_1) - u(p) + i\left[v(p+h) - v(p_1+h_1) + v(p+h_1) - v(p)\right]$$

$$\begin{aligned} = \; & u(p_1+h_1, p_2+h_2) - u(p_1+h_1, p_2) \\ & + u(p_1+h_1, p_2) - u(p_1, p_2) \\ & + i\left[v(p_1+h_1, p_2+h_2) - v(p_1+h_1, p_2)\right] \\ & + i\left[v(p_1+h_1, p_2) - v(p_1, p_2)\right] \end{aligned}$$

apply MVT 4-times

$$= \frac{\partial u}{\partial y}(p_1+h_1, p_2+h_2^*)h_2 + \frac{\partial u}{\partial x}(p_1+h_1^*, p_2)h_1 + i\left[\frac{\partial v}{\partial y}(p_1+h_1, p_2+h_2^{**})h_2 + \frac{\partial v}{\partial x}(p_1+h_1^{**}, p_2)h_1\right]$$

The calculation on the last page helps motivates our construction of $L = df_p$ (we hope to prove this assertion)

$$L(h) = h_1 L(1) + h_2 L(i)$$

$$L(h) = h_1 \frac{\partial f}{\partial x}(p) + h_2 \frac{\partial f}{\partial y}(p)$$

(Claim, this definition for $L$ "works")

To show $L$ works,

① $$L(ch+k) = (ch+k)_1 \frac{\partial f}{\partial x}(p) + (ch+k)_2 \frac{\partial f}{\partial y}(p)$$
$$= c\left(h_1 \frac{\partial f}{\partial x}(p) + h_2 \frac{\partial f}{\partial y}(p)\right) + \left(k_1 \frac{\partial f}{\partial x}(p) + k_2 \frac{\partial f}{\partial y}(p)\right)$$
$$= cL(h) + L(k) \quad \therefore \ \underline{L \text{ is linear}}. \checkmark$$

② We need to calculate $\Delta f - L(h)$,

$$\Delta f - L(h) = \overbrace{\left(\frac{\partial u}{\partial x}(p_1+h_1^*, p_2) + i\frac{\partial v}{\partial x}(p_1+h_1^{**}, p_2) - \frac{\partial u}{\partial x}(p) - i\frac{\partial v}{\partial x}(p)\right)}^{\alpha(h)} h_1 +$$
$$+ \underbrace{\left(\frac{\partial u}{\partial y}(p_1+h_1, p_2+h_2^*) + i\frac{\partial v}{\partial y}(p_1+h_1, p_2+h_2^{**}) - \frac{\partial u}{\partial y}(p) - i\frac{\partial v}{\partial y}(p)\right)}_{\beta(h)} h_2$$

[By continuity of the partial derivatives, note as $h \to 0$ so likewise $h_1^*, h_2^*, h_1^{**}, h_2^{**} \to 0$ and hence $\alpha(h) \to 0$ and $\beta(h) \to 0$ as $h \to 0$.]

② continued, we need to show that

$$\frac{\Delta f - L(h)}{\|h\|} = \frac{f(p+h) - f(p) - L(h)}{\|h\|} \longrightarrow 0$$

as $h = h_1 + ih_2 = (h_1, h_2) \longrightarrow 0$. Consider, $h \neq 0$

$$\frac{f(p+h) - f(p) - L(h)}{\|h\|} = \frac{\alpha(h)h_1 + \beta(h)h_2}{\|h\|}$$

Take modulus, for $h \neq 0$, notice, $\|h\| = \sqrt{h_1^2 + h_2^2} = |h_1 + ih_2|$
$= |h|$

$$0 \leq \left| \frac{f(p+h) - f(p) - L(h)}{|h|} \right| = \frac{|\alpha(h)h_1 + \beta(h)h_2|}{|h|}$$

$$\leq |\alpha(h)| \frac{|h_1|}{|h|} + |\beta(h)| \frac{|h_2|}{|h|}$$

$|h_1| \leq |h|$
$|h_2| \leq |h|$

$$\leq |\alpha(h)| \frac{|h|}{|h|} + |\beta(h)| \frac{|h|}{|h|}$$

$$= |\alpha(h)| + |\beta(h)|$$

Thus, by Squeeze Th$^{\underline{m}}$, as $|\alpha(h)| + |\beta(h)| \longrightarrow 0$ as $h \to 0$ we find $\left| \frac{f(p+h) - f(p) - L(h)}{|h|} \right| \longrightarrow 0$

Consequently, $\lim\limits_{h \to 0} \left( \frac{f(p+h) - f(p) - L(h)}{\|h\|} \right) = 0$

and we've shown $f$ is differentiable at $p$ with differential $df_p(h) = h_1 \frac{\partial f}{\partial x}(p) + h_2 \frac{\partial f}{\partial y}(p)$

$\boxed{\text{Th}^{\underline{m}}/}$ If $f = u + iv$ is continuously differentiable at $p$ then $f$ is differentiable at $p$ with differential given by: ← Frechet sense

$$df_p(h) = h_1 \frac{\partial f}{\partial x}(p) + h_2 \frac{\partial f}{\partial y}(p).$$

continuously differentiable at $p$ means partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at $p$ and are continuous on some open set containing $p$.

differentiable at $p$ means we can find a differential $df_p : \mathbb{C} \longrightarrow \mathbb{C}$ which solves the limiting Frechet Quotient

$$\lim_{h \to 0} \left( \frac{f(p+h) - f(p) - df_p(h)}{\|h\|} \right) = 0$$