Show **your** work carefully. 3pts per problem. Be sure to write both my problem number as well as the number from Dummit and Foote on your solution. Thanks!

Problem 71: Dummit and Foote, §10.1#5 (submodule proof)

Problem 72: Dummit and Foote, §10.1#8 (torsion)

Problem 73: Dummit and Foote, §10.1#9 (annihilator)

Problem 74: Dummit and Foote, §10.1#22 (on the natural characterization of R-algebras)

Problem 75: Dummit and Foote, §10.2#1 (who says I never give gifts ?)

Problem 76: Dummit and Foote, §10.2#9 (fuzzy comment: commutative ring and dual space to R-module)

Problem 77: Dummit and Foote, §10.2#10 (fuzzy comment: R-module regular representation)

Problem 78: Let A_1, A_2 be R-modules and let B_1 be a submodule of A_1 whereas B_2 is a submodule of A_2. Prove that

$$(A_1 \times A_2)/(B_1 \times B_2) \cong (A_1/B_1) \times (A_2/B_2)$$

Problem 79: Dummit and Foote, §10.3#4 (torsion module)

Problem 80: Dummit and Foote, §10.3#13 (isomorphism, looks like dual space)

Problem 81: Dummit and Foote, §10.3#15 (central idempotent)

Problem 82: Dummit and Foote, §10.4#3 (fun with $\otimes$ $\mathbb{R}$ and $\mathbb{C}$)

Problem 83: Dummit and Foote, §10.4#6 (fun with $\otimes$, field of fractions)

Problem 84: Dummit and Foote, §10.4#11 (fun with $\otimes$, simple tensor)

Problem 85: Dummit and Foote, §10.4#25 (fun with $\otimes$, isomorphism of algebras)

Problem 86: Let $\mathcal{B}$ denote the set of bilinear mappings on the vector space V over characteristic zero field F with basis $\beta = \{e_1, \dots, e_n\}$. Let β^* be the dual basis $\{e^1, \dots, e^n\}$ defined by $e^i(e_j) = \delta_{ij}$. For any $\alpha, \beta \in V^*$ we define

$$(\alpha \otimes \beta)(x, y) = \alpha(x)\beta(y)$$

for all $x, y \in V$. Furthermore, $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$

(a.) If $\mathcal{A} = \{b \in \mathcal{B} \mid b(x, y) = -b(y, x) \text{ for all } x, y \in V\}$ then find the basis and calculate the dimension of $\mathcal{A}$

(b.) Show $\mathcal{A}$ is isomorphic to $\Lambda^2(V)$ as defined in Dummit and Foote §11.5

Problem 87: Dummit and Foote, §12.1#2 (meaning of rank in module over integral domain)

Problem 88: Dummit and Foote, §12.1#3 (rank interacting with $\oplus$)

Problem 89: Dummit and Foote, §12.1#4 (rank interacting with quotient of modules)

Problem 90: Dummit and Foote, §12.1#6 (on constructing an example of rank 1 module which is not a free R-module)

Remark: exercises §12.1#15 − 20 develop the Smith Normal form calculation for the relation matrix which defines the module structure over R. We will apply the results to some examples in class as well as corresponding homework. This justifies the row and column operations I shared (and will soon share for context outside $R = F[x]$, as in we look at $R = \mathbb{Z}$ examples etc...)

Problem 91: Suppose A is a matrix for which $(x^2 - 2)^2(x - 3)^3$ is the characteristic polynomial.

- **(a.)** Find the possible invariant factors for the $\mathbb{Q}[x]$-module of A and for each case also list the elementary divisors in $\mathbb{Q}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A
- **(b.)** Find the possible invariant factors for the $\mathbb{R}[x]$-module of A and for each case also list the elementary divisors in $\mathbb{R}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A

Problem 92: Suppose A is a matrix for which $(x^3 - 2)^2(x - 3)^2$ is the characteristic polynomial.

- **(a.)** Find the possible invariant factors for the $\mathbb{Q}[x]$-module of A and for each case also list the elementary divisors in $\mathbb{Q}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A
- **(b.)** Find the possible invariant factors for the $\mathbb{R}[x]$-module of A and for each case also list the elementary divisors in $\mathbb{R}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A
- **(c.)** Find the possible invariant factors for the $\mathbb{C}[x]$-module of A and for each case also list the elementary divisors in $\mathbb{C}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A

Problem 93: Suppose $G = \{(a, b, c) \in \mathbb{Z}^3 \mid 2a + 4b + 4c = 0, 6b + 6c = 0, 8c = 0\}$. Find the decomposition of G as a $\mathbb{Z}$-module in invariant factor form as well as elementary divisor form.

Problem 94: Dummit and Foote, §12.3#17 (every square matrix similar to its transpose)

Problem 95: Dummit and Foote, §12.3#22 (diagonalizability of matrix over $\mathbb{C}$)

Problem 96: Let V be a vector space of dimension n over the field F. Suppose E is an extension field of F for which $\delta = \{w_1, \dots, w_m\}$ serves as an F-basis of E. Let $\beta = \{v_1, \dots, v_n\}$ form an F-basis for V. We wish to study the E-vector space $E \otimes_F V$ in this problem. Notice,

$$c(a \otimes v) = (ca) \otimes v$$

defines the scalar multiplication of E on $E \otimes_F V$. Prove the following:

- **(a.)** If $\gamma = 1 \otimes \beta = \{1 \otimes v_1, \dots, 1 \otimes v_n\}$ then γ is an E-basis for $E \otimes_F V$. Notice by the theory developed in Dummit and Foote we know $\{w_i \otimes v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ serves as an F-basis for $E \otimes_F V$.
- **(b.)** Suppose $T : V \to V$ is a linear transformation. Let $T_E : E \otimes_F V \to E \otimes_F V$ be defined by $T_E(c \otimes x) = c \otimes T(x)$ for all $c \otimes x \in E \otimes_F V$. Show T_E is a linear transformation on $E \otimes_F V$ and show that $[T]_{\beta,\beta}$ and $[T_E]_{\gamma,\gamma}$ are related.

Problem 97: The complexification of a $2k$-dimensional vector space V over $\mathbb{R}$ is given by $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$. We suppress explicit $\otimes$ notation via the following conventions: for $x \in V$ and $a + ib \in \mathbb{C}$,

$$1 \otimes x = x \qquad \& \qquad i \otimes x = ix$$

hence $(a + ib) \otimes x = (a + ib)x = ax + ibx$ and $T_{\mathbb{C}}(x + iy) = T(x) + iT(y)$ for all $x, y \in V$ defines $T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$.

(a.) Suppose $\Gamma = \{v_1, \dots, v_k\}$ is a k-chain with eigenvalue $\lambda = \alpha + i\beta$ for $T_{\mathbb{C}}$. Let $v_j = a_j + ib_j$ for $j = 1, \dots, k$ where $a_j, b_j \in V$. Prove $\Upsilon = \{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$ is $\mathbb{R}$-linearly independent given that $\beta \neq 0$.

(b.) Extend Γ to a basis δ for $V_{\mathbb{C}}$ such that $[T_{\mathbb{C}}]_{\delta,\delta}$ is a Jordan form

(c.) Show $[T]_{\Upsilon,\Upsilon} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \otimes I_k + I_2 \otimes N_k$

Problem 98: Consider a real associative unital algebra $\mathcal{A}$ of finite dimension with multiplication $\star$. If $\mathcal{R}_{\mathcal{A}}$ is the set of linear transformations on $\mathcal{A}$ for which $T(x \star y) = T(x) \star y$ for all $x, y \in \mathcal{A}$ then prove $\mathcal{R}_{\mathcal{A}}$ is real algebra which is isomorphic to $\mathcal{A}$.

Problem 99: Consider a real associative unital algebra $\mathcal{A}$ of finite dimension n. Prove there exists a matrix subalgebra of $\mathbb{R}^{n \times n}$ which is isomorphic to $\mathcal{A}$.

Problem 100: Consider a real associative unital algebra $\mathcal{A}$ of finite dimension n. Prove every element of $\mathcal{A}$ is either a zero, a unit, or a zero-divisor.

Problem 101: Find the matrix regular representation for each real associative algebra given below:

(a.) $\mathbb{C} \times \mathbb{C}$

(b.) $\mathcal{H} \times \mathbb{C}$

(c.) $\mathbb{R}^{2\times 2}$

Find the group of units for each algebra above.

Problem 102: Let $\mathcal{A}$ and $\mathcal{B}$ be real associative algebras of finite dimension and suppose $M_{\beta_1}(\mathcal{A})$ and $M_{\beta_2}(\mathcal{B})$ are the representations of $\mathcal{A}$ and $\mathcal{B}$ with respect to bases β_1, β_2. Show that $\beta = (\beta_1 \times 0) \cup (0 \times \beta_2)$ basis for $\mathcal{A} \times \mathcal{B}$ has

$$M_\beta(\mathcal{A} \times \mathcal{B}) = M_{\beta_1}(\mathcal{A}) \oplus M_{\beta_2}(\mathcal{B}).$$

Problem 103: The algebras below are all 9-dimensional over $\mathbb{R}$. Determine which algebras are isomorphic and either provide the isomorphism or give evidence that no isomorphism can be found.

$$\mathbb{R}^{3\times 3} \qquad \mathcal{H}_9 = \mathbb{R} \oplus j\mathbb{R} \oplus \cdots \oplus j^8\mathbb{R} \qquad \mathcal{C}_9 = \mathbb{R} \oplus k\mathbb{R} \oplus \cdots \oplus k^8\mathbb{R} \qquad \Gamma_9 = \mathbb{R} \oplus \varepsilon\mathbb{R} \oplus \cdots \oplus \varepsilon^8\mathbb{R}$$

where $j^9 = 1$ and $k^9 = -1$ and $\varepsilon^9 = 0$.

Problem 104: The algebras below are all 4-dimensional over $\mathbb{R}$. Determine which algebras are isomorphic and either provide the isomorphism or give evidence that no isomorphism can be found.

(i.) the matrix algebra $\mathbb{R}^{2\times 2}$

(ii.) $\mathcal{H} \times \mathcal{H}$ where $\mathcal{H}$ is the hyperbolic numbers

(iii.) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^4$, the direct product algebra

(iv.) $\mathbb{R} \times \mathcal{H}_3$ the direct product of $\mathbb{R}$ with the 3-hyperbolic numbers

(v.) $\mathbb{C} \times \mathcal{H}$

Problem 105: If we think about a commutative associative algebra $\mathcal{A}$ of dimension n and we form $\mathcal{M} = \mathcal{A}^{2\times 2}$ then $\mathcal{M}$ is naturally a non-commutative associative algebra. Find a matrix representation of $\mathcal{M}$ in $\mathbb{R}^{4n\times 4n}$. Describe the units of $\mathcal{M}$.