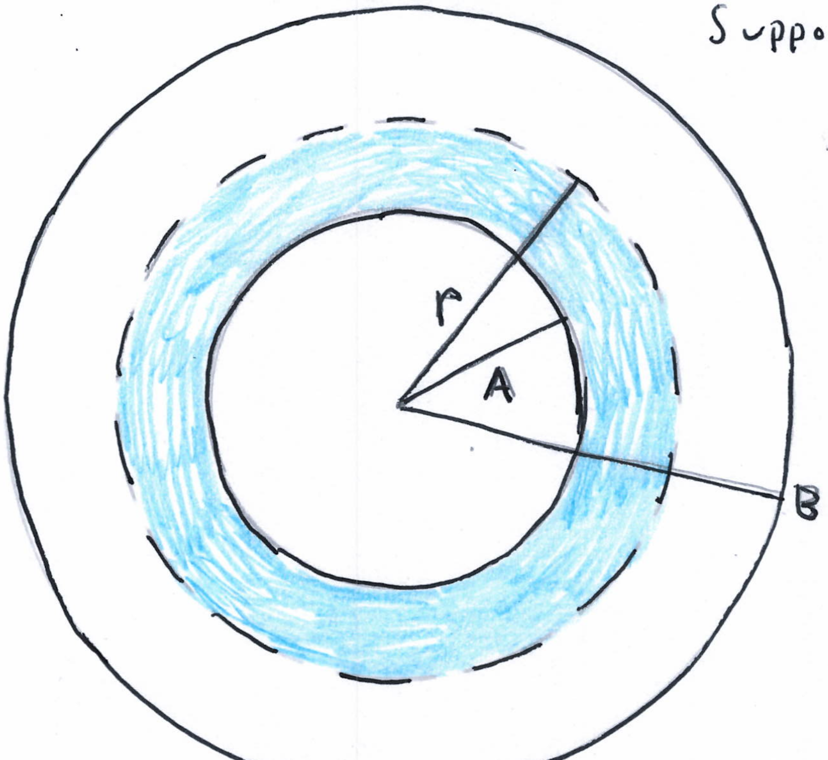
Print Names in Team: ______________________________

PHYSICS 232	MISSION 2: GAUSS' LAW & ELECTRIC FLUX

Please work each problem in the white space provided. Attach additional sheets if necessary. Print this one-sided and staple in the top left corner with a metal staple once complete. Each team turns in one document.

Problem 7 A uniform charge Q is evenly distributed over a spherical shell with inner radius A and outer radius B. Find the magnitude of the electric field as function of the distance r from the center of the spherical shell.



Suppose $A < r < B$ notice,

$$\rho = \frac{Q}{\frac{4}{3}\pi(B^3 - A^3)}$$

$$Q_{enc} = \rho \, \underline{Vol_{enc}} = \rho \cdot \frac{4}{3}\pi(r^3 - A^3) = \frac{Q(r^3 - A^3)}{B^3 - A^3}$$

By the spherical symmetry of the given charge distribution and Gauss' Law,

$$\frac{Q_{enc}}{\epsilon_0} = \Phi_E \hookrightarrow \frac{Q(r^3 - A^3)}{\epsilon_0(B^3 - A^3)} = 4\pi r^2 E$$

$$\therefore \boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}\left(\frac{r^3 - A^3}{B^3 - A^3}\right)} \leftarrow A \leq r \leq B$$

(radially outward)

Likewise, by Gauss' Law for $r < A$ we find $Q_{enc} = 0$ and hence $\boxed{E = 0 \text{ for } 0 \leq r \leq A}$

Finally if $r \geq B$ then $Q_{enc} = Q$ thus

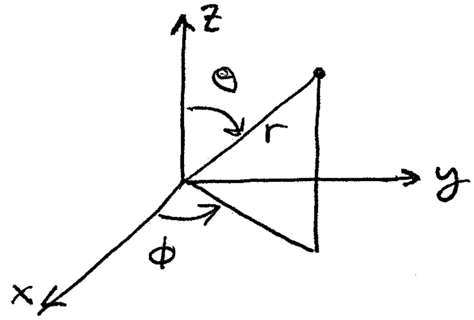
$$\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}} \xleftarrow{r \geq B}$$

(radially outwards)

Problem 8 Suppose $\vec{E}(x,y,z) = \langle y, 3x, z\rangle$ is the electric field due to some unknown distribution of charge. Find the flux of this field through the unit-sphere ($x^2+y^2+z^2=1$). What is the net-charge enclosed ? (recall I showed how to calculate flux in Week 1's lectures, you may omit units for this problem)

Spherical coordinates r, θ, ϕ with $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$



Parametrize sphere by setting $r = R$ thus

$$\left.\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}\right\} \quad \vec{r}(\theta,\phi) = \langle R\sin\theta\cos\phi,\ R\sin\theta\sin\phi,\ R\cos\theta\rangle$$

Hence, $\dfrac{\partial\vec{r}}{\partial\theta} = R\langle\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta\rangle$

$\dfrac{\partial\vec{r}}{\partial\phi} = R\langle -\sin\theta\sin\phi,\ \sin\theta\cos\phi,\ 0\rangle$

Thus $\dfrac{\partial\vec{r}}{\partial\phi}\times\dfrac{\partial\vec{r}}{\partial\theta} = R^2\sin\theta\langle -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta\rangle$

Oops, I should calculate $\dfrac{\partial\vec{r}}{\partial\theta}\times\dfrac{\partial\vec{r}}{\partial\phi} = R^2\sin\theta\langle\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\rangle$

Thus $d\vec{A} = R^2\sin\theta\langle\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\rangle\, d\theta\, d\phi$

Calculate $\displaystyle\int_{S_R}\vec{E}\cdot d\vec{A}$ we evaluate $\vec{E}$ at $\vec{r}(\theta,\phi)$, $\underset{\text{given}}{\underline{R=1}}$

$$\int_{S_R}\vec{E}\cdot d\vec{A} = \int_0^{\pi}\int_0^{2\pi}\langle\sin\theta\sin\phi,\ 3\sin\theta\cos\phi,\ \cos\theta\rangle\cdot d\vec{A}$$

$$= \int_0^{\pi}\int_0^{2\pi}\sin\theta\left(\sin^2\theta\,\overbrace{\cancel{\sin\phi\cos\phi}}^{0} + 3\sin^2\theta\,\overbrace{\cancel{\sin\phi\cos\phi}}^{0} + \cos^2\theta\right)d\phi\, d\theta$$

$$= 2\pi\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta \qquad \boxed{\begin{aligned} u &= \cos\theta & u(0) &= 1 \\ du &= -\sin\theta\, d\theta & u(\pi) &= -1 \end{aligned}}$$

$$= 2\pi\int_1^{-1}u^2(-du) = 4\pi\int_0^1 u^2\, du = \boxed{\frac{4\pi}{3}}$$

P8 continued

So I showed how to calculate

$\Phi_E = \int_{S_R} \vec{E} \cdot d\vec{A}$ explicitly on the last page

and since $\Phi_E = \frac{Q_{enc}}{\epsilon_0}$ we deduce

$$\boxed{Q = \frac{4\pi\epsilon_0}{3}}$$

As I mentioned in lecture, there is a better way, the DIVERGENCE THEOREM

$$\iint_{S_R = \partial B_R} \vec{E} \cdot d\vec{A} = \iiint_{B_R} (\nabla \cdot \vec{E})\, dV$$

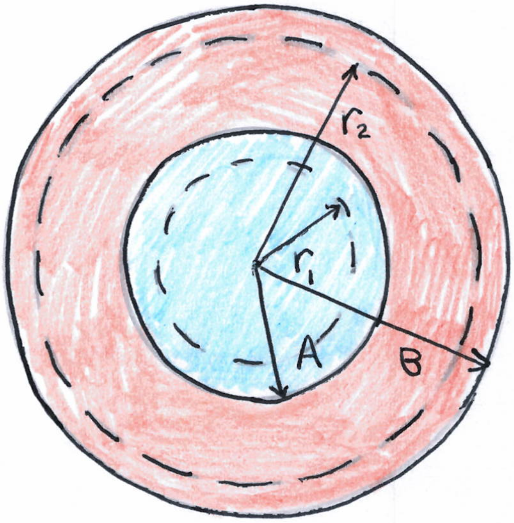
$$= \iiint_{B_R} \Big(\underbrace{\frac{\partial}{\partial x}(y)}_{0} + \underbrace{\frac{\partial}{\partial y}(3x)}_{0} + \underbrace{\frac{\partial}{\partial z}(z)}_{1} \Big)\, dV$$

$$= \iiint_{B_R} dV \qquad (\text{But } R = 1)$$

$$= \frac{4}{3}\pi \hookrightarrow \underline{Q = \frac{4\pi\epsilon_0}{3}}.$$

Remark: the divergence of $\vec{E}$ is only nonzero where field lines begin or end.
As we derived $\boxed{\nabla \cdot \vec{E} = \rho/\epsilon_0}$ is the differential form of Gauss' Law.

Problem 9 Suppose a charge Q_1 is uniformly distributed over a spherical region up to radius $r = A$. Then a charge Q_2 is uniformly distributed over $A \leq r \leq B$. Calculate the magnitude of the electric field as a function of the radius.



$$0 \leq r_1 \leq A$$

$$A \leq r_2 \leq B$$

$$\underbrace{\rho_1 = \frac{Q_1}{\frac{4}{3}\pi A^3}}_{0 \leq r < A} \quad \& \quad \underbrace{\rho_2 = \frac{Q_2}{\frac{4}{3}\pi (B^3 - A^3)}}_{A < r \leq B}$$

- Gauss' Law for $0 \leq r_1 < A$ applied to sphere of radius r_1,

$$\frac{Q_{enc}}{\epsilon_0} = \frac{\rho_1 \frac{4}{3}\pi r_1^3}{\epsilon_0} = \Phi_E = 4\pi r_1^2 E$$

$$\Rightarrow \quad E = \frac{Q_1 r_1^3}{4\pi\epsilon_0 r_1^2 A^3}$$

$$\boxed{E = \frac{Q_1 r}{4\pi\epsilon_0 A^3}} \quad \begin{array}{l} (0 \leq r \leq A) \\ (\text{radially outward}) \end{array}$$

- Gauss' Law for $A < r_2 < B$ applied to sphere of radius r_2,

$$\frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\left(Q_1 + \frac{Q_2 (r_2^3 - A^3)}{B^3 - A^3}\right) = \Phi_E = 4\pi r_2^2 E$$

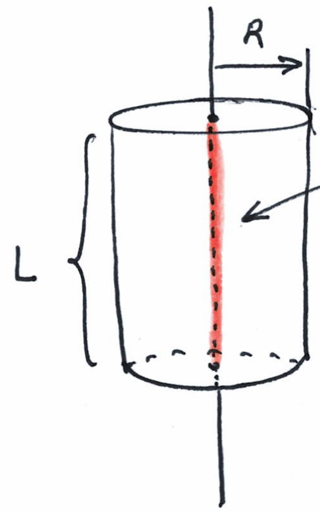
Replacing r_2 with r,

$$\boxed{E = \frac{1}{4\pi\epsilon_0 r^2}\left(Q_1 + Q_2\left(\frac{r^3 - A^3}{B^3 - A^3}\right)\right)} \quad \begin{array}{l} (A \leq r < B) \\ (\text{radially outward}) \end{array}$$

Lastly, if $r \geq B$ then $Q_{enc} = Q_1 + Q_2$ for a sphere of radius r hence

$$\boxed{E = \frac{Q_1 + Q_2}{4\pi\epsilon_0 r^2}} \quad \begin{array}{l} (r \geq B) \\ (\text{radially outward}) \end{array}$$

Problem 10 Suppose a flux of 3.0 Nm^2/C is measured to cross a cylinder of radius 5.0 cm and length $L = 30.0\ cm$ which encircles a long wire placed at the center of the cylinder. What is the charge per unit length in the wire?



$$Q_{enc} = \lambda L$$

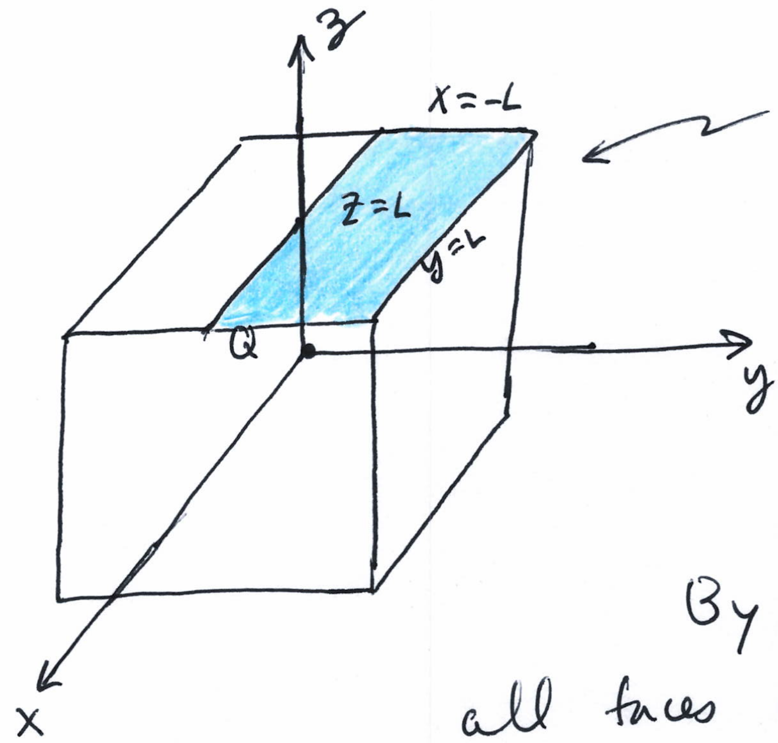
$$\Phi_E = 2\pi R L E = 3.0\ \frac{Nm^2}{C}$$

$$\frac{Q_{enc}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0} = 3.0\ \frac{Nm^2}{C}$$

$$\lambda = \frac{(3.0)(8.8542 \times 10^{-12})}{0.300}\ \frac{C}{m}$$

$$\boxed{\lambda = 8.854 \times 10^{-11}\ \frac{C}{m}}$$

Problem 11 Suppose a charge Q is placed at the origin. Find the flux through the region given by $-L \le x \le L$ at $z = L$ for $0 \le y \le L$.



$\frac{1}{12}$ of the total area on the surface of the cube with $-L \le x, y, z \le L$

By symmetry, since all faces are same relative to the origin where Q is placed,

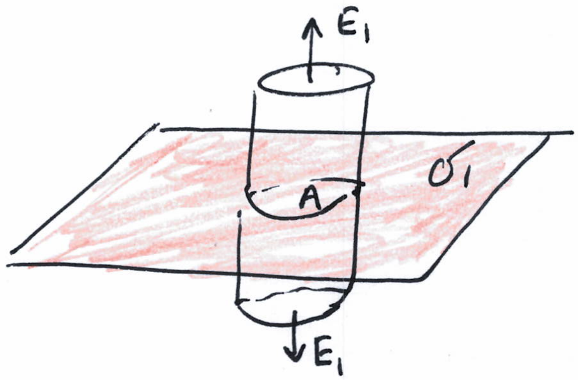
$$\Phi_{\text{region}} = \frac{1}{12} \Phi_{\text{TOTAL}} = \frac{1}{12}\left(\frac{Q}{\epsilon_0}\right)$$

$$\therefore \boxed{\Phi_R = \frac{Q}{12\epsilon_0}}$$

Problem 12 Suppose we have two very large planes with charge density $\sigma_1 > 0$ at $z = z_1$ and a second with charge density $\sigma_2 = 2\sigma_1$ at $z = z_2$ where $z_1 < z_2$. Derive the electric field in each region via an appropriate application of Gauss' Law:

(a.) $z < z_1$

I explained in class, we should use superposition



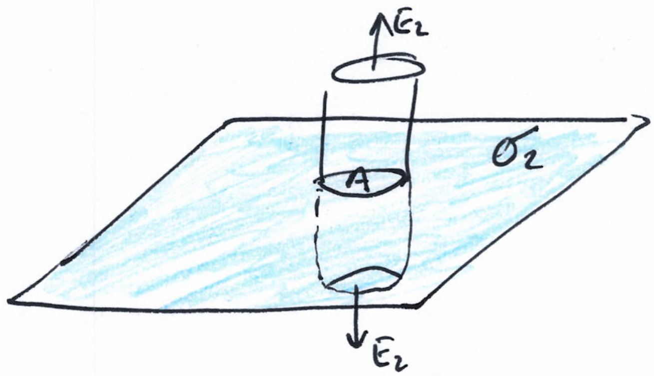
$$\Phi_{E_1} = \frac{Q_{enc}}{\epsilon_0}$$

$$2E_1 A = \frac{\sigma_1 A}{\epsilon_0}$$

$$E_1 = \frac{\sigma_1}{2\epsilon_0} \text{ away from plane}$$

- Likewise for E_2, from σ_2

(b.) $z_1 < z < z_2$

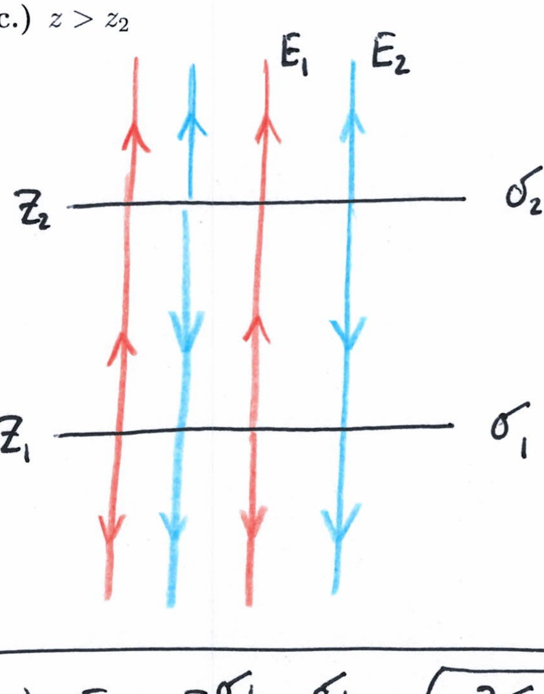


$$\Phi_{E_2} = \frac{Q_{enc}}{\epsilon_0}$$

$$2E_2 A = \frac{\sigma_2 A}{\epsilon_0}$$

$$E_2 = \frac{\sigma_2}{2\epsilon_0} \text{ away from plane.}$$

(c.) $z > z_2$



$$E_1 = \frac{\sigma_1}{2\epsilon_0} : \text{away from } Z_1$$

$$E_2 = \frac{2\sigma_1}{2\epsilon_0} = \frac{\sigma_1}{\epsilon_0} : \text{away from } Z_2$$

By superposition,

(a.) $E = \frac{\sigma_1}{2\epsilon_0} + \frac{\sigma_1}{\epsilon_0} = \boxed{\frac{3\sigma_1}{2\epsilon_0}}$ up

(b.) $E = \frac{\sigma_1}{2\epsilon_0} - \frac{\sigma_1}{\epsilon_0} = \boxed{\frac{-\sigma_1}{2\epsilon_0}}$ up

or, perhaps better $\boxed{\frac{\sigma_1}{2\epsilon_0} \text{ down}}$

(c.) $E = \frac{-\sigma_1}{2\epsilon_0} - \frac{\sigma_1}{\epsilon_0} = \boxed{\frac{-3\sigma_1}{2\epsilon_0} \text{ up}}$ or $\boxed{\frac{3\sigma_1}{2\epsilon_0} \text{ down}}$