Print Names in Team: ______________________________

PHYSICS 232 — MISSION 3: POTENTIAL & CAPACITANCE

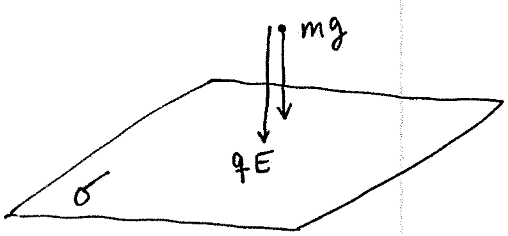
Please work each problem in the white space provided. Attach additional sheets if necessary. Print this one-sided and staple in the top left corner with a metal staple once complete. Each team turns in one document.

Problem 13 A charge q is thrown vertically (on Earth) with speed v_o and it reaches a maximum height H (in the absence of any external electric field). Next, a horizontal plane made from an insulator is charged with a positive charge-density σ. The charge q is thrown vertically with speed v_o once again but only reaches a height $H/2$. If the mass of the charge is m then find σ in terms of the given charge, mass and perhaps g.

Energy conservation: $\frac{1}{2}mv_o^2 = mgH \quad \therefore \quad \underline{v_o^2 = 2gH}$.

Gauss' Law $\Rightarrow E = \frac{\sigma}{2\epsilon_o}$ directed away from plane directly $\perp$

otherwise motion impossible. ($q < 0$)



$ma = -mg + qE$

$a = -g + \frac{qE}{m}$ Constant acc.

$v_f^2 = v_o^2 + 2a\Delta y \Rightarrow -v_o^2 = 2a\left(\frac{H}{2}\right)$

$\Rightarrow -2gH = aH$

$\Rightarrow a = -2g$

$\Rightarrow -g + \frac{qE}{m} = -2g$

$\Rightarrow \frac{qE}{m} = -g$

$\Rightarrow \frac{q\sigma}{2\epsilon_o m} = -g \quad \therefore \quad \boxed{\sigma = \frac{-2m\epsilon_o g}{q}}$

Remark: could also use electric potential energy to solve this problem.

$PE = mgy + \left(\frac{\sigma q}{2\epsilon_o}\right)(-y)$

$\frac{1}{2}mv_o^2 = mgH = mg\left(\frac{H}{2}\right) + \frac{\sigma q}{2\epsilon_o}\frac{H}{2}$

$\rightarrow -\frac{mgH}{2} = \frac{\sigma q H}{4\epsilon_o}$

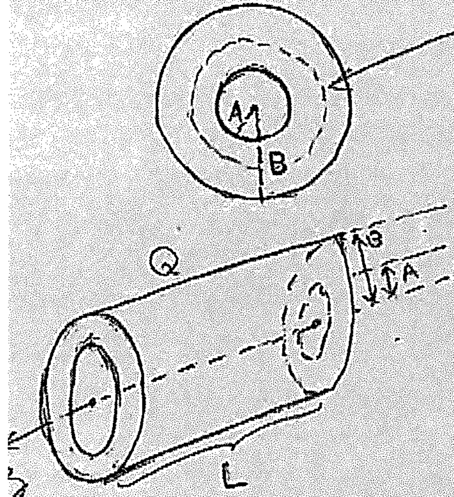
$\boxed{\sigma = \frac{-2m\epsilon_o g}{q}}$

PROBLEM 14

: Suppose a very long line of charge with Q per length L is distributed evenly along the z-axis from inner radius A to outer radius B. Find the potential for this charge distribution as a function of a distance r from the z-axis. ~~Assume the potential is set to zero as $r \to \infty$~~ ← crazy.

(z-axis out of page for diagram below)

By symmetry, $\vec{E} = E\hat{r}$



① imagine Gaussian cylinder, length L radius r, with $A \le r \le B$,

$$\Phi = Q_{enc}/\epsilon_0 \quad , \quad \rho = \frac{Q}{\pi L(B^2-A^2)}$$

$$2\cancel{\pi} r \cancel{L} E = \left(\frac{Q}{\pi L(B^2-A^2)}\right)\left(\cancel{\pi}(r^2-A^2)\cancel{L}\right)\frac{1}{\epsilon_0}$$

$$E = \frac{Q(r^2-A^2)}{2\pi r L(B^2-A^2)\epsilon_0} = \frac{\alpha(r^2-A^2)}{r}$$

(for $A \le r \le B$)

② Note, for $r \le A$ we have $E = 0$.

③ For $r \ge B$ the $Q_{enc} = Q$ for a length L cylinder,

$$2\pi r L E = Q/\epsilon_0 \Rightarrow E = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{r}$$

(note: $\lambda = \frac{Q}{L}$ may make this familiar.)

Now to find V we may integrate $E = -\frac{dV}{dr}$.

Setting $V(r) = 0$ (gauge choice)

we find $V(r) = 0$ for $0 \le r \le A$

$$E = \alpha\left(r - \frac{A^2}{r}\right) = \frac{-dV}{dr} \Rightarrow V(r) = -\alpha\left(\tfrac{1}{2}r^2 - A^2\ln(r)\right) + C$$

Need to choose C to fit $V(A) = 0$.

$$V(A) = \left(-\alpha \left(\tfrac{1}{2} r^2 - A^2 \ln(r) \right) + C \right) \Big|_{r=A}$$

$$\Rightarrow \quad 0 = -\alpha \left(\frac{A^2}{2} - A^2 \ln(A) \right) + C$$

$$\Rightarrow \quad C = \alpha \left(\tfrac{1}{2} A^2 - A^2 \ln(A) \right) \qquad \text{where } \alpha = \frac{Q}{2\pi L (B^2 - A^2) \epsilon_0}$$

$$\Rightarrow V(r) = \alpha \left(A^2 \ln(r) - \tfrac{1}{2} r^2 + \tfrac{1}{2} A^2 - A^2 \ln(A) \right)$$

$$\boxed{V(r) = \frac{Q}{2\pi L (B^2 - A^2) \epsilon_0} \underbrace{\left(A^2 \ln\left(\frac{r}{A}\right) - \frac{1}{2}(r^2 - A^2) \right)}_{\circledast}}$$

for $A \le r \le B$

Integrating $E = \dfrac{Q}{2\pi \epsilon_0 L} \dfrac{1}{r} = -\dfrac{dV}{dr}$ for $r \ge B$

yields,
$$V(r) = \frac{-Q}{2\pi \epsilon_0 L} \ln(r) + C_2$$

We need $V(B) = \dfrac{-Q}{2\pi \epsilon_0 L} \ln(B) + C_2 = \circledast \Big|_{r=B}$

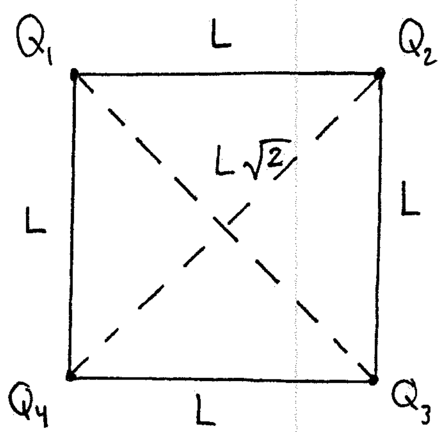
$$\frac{-Q}{2\pi \epsilon_0 L} \ln(B) + \underline{\underline{C_2}} = \alpha \left(A^2 \ln\left(\frac{B}{A}\right) - \frac{1}{2} (B^2 - A^2) \right)$$

$$\boxed{V(r) = \frac{Q}{2\pi \epsilon_0 L} \left[\ln\left(\frac{B}{r}\right) + \frac{1}{B^2 - A^2} \left(A^2 \ln\left(\frac{B}{A}\right) - \frac{1}{2}(B^2 - A^2) \right) \right]}$$

for $r \ge B$. Recall we mentioned last page $V(r) = 0$ for $0 \le r \le A$.

done

Problem 15 Suppose that 4 particles with a charge 2.0 Coulombs each are assembled in a square pattern. If the side-length of the square is 1.0 meters then what is the energy required to create this charge configuration?



I.) $U_1 = 0$ to place Q_1 in empty space.

II.) $U_2 = \dfrac{kQ_1Q_2}{L}$ to place Q_2 alongside Q_1 as pictured

III.) $U_3 = U_2 + \dfrac{kQ_1Q_3}{L\sqrt{2}} + \dfrac{kQ_2Q_3}{L}$ to assemble Q_1, Q_2, Q_3

IV.) $U_4 = U_3 + \dfrac{kQ_1Q_4}{L} + \dfrac{kQ_2Q_4}{L\sqrt{2}} + \dfrac{kQ_3Q_4}{L}$ to place Q_1, Q_2, Q_3, Q_4

Then setting $Q_1 = Q_2 = Q_3 = Q_4 = Q = 2.0\,C$, $L = 1.0\,m$,

$$U_4 = \frac{kQ^2}{L}\left(1 + \frac{1}{\sqrt{2}} + 1 + 1 + \frac{1}{\sqrt{2}} + 1\right)$$

$$= \frac{kQ^2}{L}\left(4 + \sqrt{2}\right) = \frac{Q^2}{4\pi\epsilon_0 L}\left(4+\sqrt{2}\right)$$

$$= \left[\frac{(8.99\times 10^9)(2.0)^2}{1.0}\right]\left(4+\sqrt{2}\right) J$$

$$= \frac{4.0}{4\pi\ 8.854\times 10^{-12}}\left(4+\sqrt{2}\right) J$$

$$= \boxed{1.946\times 10^{11}\ J}$$

Problem 16 Suppose $V(x, y, z) = \alpha(x^2 + y^2) + \beta z$ where α, β are given constants. Find the following:

(a.) The electric field at (x, y, z),

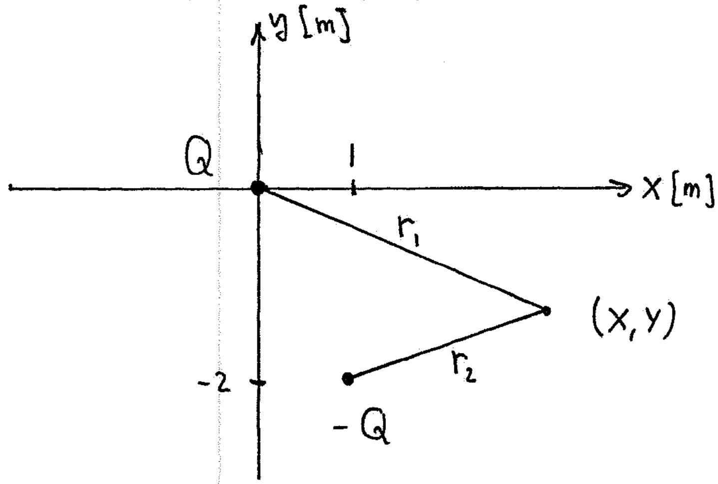
$$\vec{E} = -\nabla V = \left\langle -\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z} \right\rangle$$

$$\Rightarrow \quad \boxed{\vec{E} = \langle -2\alpha x, -2\alpha y, -\beta \rangle}$$

(b.) The work required to move a charge $Q = 2.0\ \mu C$ from the origin to $(1, 2, 2)m$.

$$\begin{aligned} W = \Delta KE &= -\Delta PE \\ &= -Q\Delta V \\ &= -Q\left(V((1,2,2)m) - V(0,0,0)\right) \\ &= -Q\left(\alpha\left((1m)^2 + (2m)^2\right) + \beta(2m)\right) - 0) \\ &= (-2.0\mu C)\left[5m^2\alpha + 2m\beta\right] \\ &= \boxed{-10\mu Cm^2\alpha - 4\mu Cm\beta} \end{aligned}$$

Problem 17 If a charge of Q is placed at the origin and another charge $-Q$ is placed at $(1, -2)m$ then find the electric potential at (x, y). Assume the potential is set to zero at infinity.



$$r_1 = \sqrt{x^2 + y^2}$$
$$r_2 = \sqrt{(x - 1m)^2 + (y + 2m)^2}$$

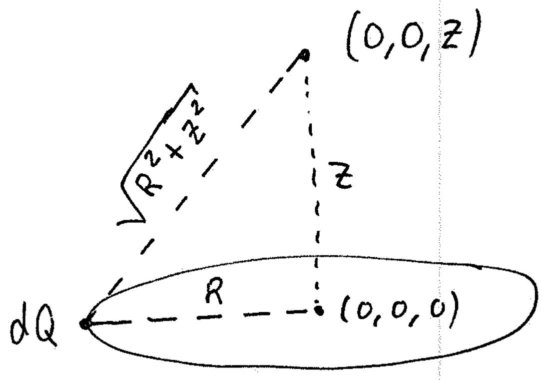
$$V(x,y) = \frac{kQ}{r_1} + \frac{k(-Q)}{r_2}$$

$$\boxed{V(x,y) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2}}}\right)}$$

$x_0 = 1m$
$y_0 = -2m$

$$r_2 = \underbrace{\sqrt{(x - 1m)^2 + (y + 2m)^2}}_{\text{ugly.}}$$

Problem 18 Suppose a ring of radius R and charge Q is set in the xy-plane centered at the origin. Find the electric potential and electric field at $(0, 0, z)$.



$$\underbrace{dV = \frac{k\,dQ}{\sqrt{R^2+z^2}}}_{\text{same for each } dQ \text{ all around the ring.}}$$

$$V = \int_{ring} dV = \int_{ring} \frac{k\,dQ}{\sqrt{R^2+z^2}} = \frac{k}{\sqrt{R^2+z^2}} \int_{ring} dQ = \frac{kQ}{\sqrt{R^2+z^2}}$$

$$\therefore \boxed{V(0,0,z) = \frac{kQ}{\sqrt{R^2+z^2}} = \frac{Q}{4\pi\epsilon_0\sqrt{R^2+z^2}}}$$

$$E_z = -\frac{dV}{dz} = \frac{-Q}{4\pi\epsilon_0}\frac{d}{dz}\left[\frac{1}{\sqrt{R^2+z^2}}\right] = \frac{-Q}{4\pi\epsilon_0}\left(\frac{-1}{2}(R^2+z^2)^{-3/2}\cdot(2z)\right)$$

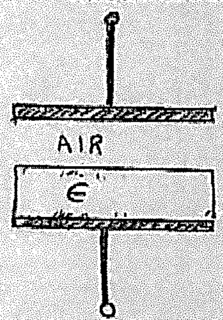
$$E_z = \frac{Q}{4\pi\epsilon_0}\left[\frac{z}{(R^2+z^2)^{3/2}}\right] = \frac{kzQ}{(R^2+z^2)^{3/2}}$$

$$\boxed{\vec{E} = \langle \underbrace{0, 0}_{\text{by symmetry of ring w.r.t. pt. } (0,0,z).}, \frac{kzQ}{(R^2+z^2)^{3/2}} \rangle} \quad (\text{at } (0,0,z))$$

[redacted] [redacted]

[redacted]

P19 [redacted] Suppose a linear dielectric material has $\epsilon = 3\epsilon_0$. If an air-filled parallel plate capacitor with original capacitance 0.5 μF is modified as to have half of the plate-to-plate volume filled with the material of dielectric ϵ then what is the new capacitance?

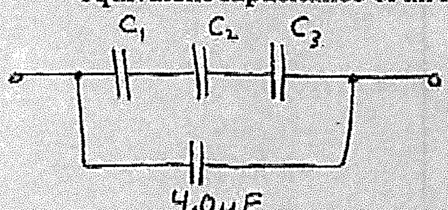


$$C_o = \frac{A\epsilon_o}{d} \Rightarrow \frac{1}{C_o} = \frac{d}{A\epsilon_o}$$

$$C_{new} = \frac{1}{\frac{1}{C_\epsilon} + \frac{1}{C_{air}}} = \left(\frac{d}{2A\epsilon} + \frac{d}{2A\epsilon_o}\right)^{-1} = \left(\frac{d}{A\epsilon_o}\left(\frac{1}{6} + \frac{1}{2}\right)\right)^{-1} = \frac{A\epsilon_o}{d}\cdot\frac{3}{2}$$

$$\therefore\ C_{new} = (1.5)(0.5\mu F) = \boxed{0.75\mu F}$$

P20 [redacted] Suppose capacitors $C_1 = 1.0\ \mu F$, $C_2 = 2.0\ \mu F$ and $C_3 = 3.0\ \mu F$ are placed in series and then a 4.0 μF capacitor is placed in parallel with the series connection of C_1, C_2, C_3. What is the equivalent capacitance of all four capacitors in the given configuration?



$$C_{eq} = 4.0\mu F + \frac{1}{\frac{1}{1.0\mu F} + \frac{1}{2.0\mu F} + \frac{1}{3.0\mu F}}$$

$$\boxed{C_{eq} = 4.545\mu F}$$

P21 [redacted] If 20 volts are applied to the capacitor configuration of the last problem then what is the current following through the circuit after a long time and what is the voltage across C_1, C_2, C_3 and C_4? Let us denote the voltage drops by V_1, V_2, V_3 and V_4 respective.

Current = 0.

$V_4 = 20V$.

$Q_1 = Q_2 = Q_3$ (in series)

$$\underbrace{C_1 V_1 = C_2 V_2 = C_3 V_3}$$

$$V_1 = 2V_2 = 3V_3$$

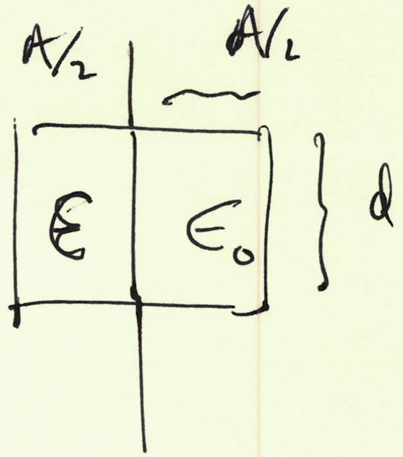
$$\Rightarrow V_2 = \frac{1}{2}V_1 \ \&\ V_3 = \frac{1}{3}V_1$$

Note: $V_1 + V_2 + V_3 = V_1 + \frac{1}{2}V_1 + \frac{1}{3}V_1 = \frac{11}{6}V_1 = 20V$

Thus $\boxed{V_1 = 10.91V} \Rightarrow \boxed{V_2 = 5.454V} \Rightarrow \boxed{V_3 = 3.636V}$

P19

(other interpretation)



$$C = \frac{A\epsilon_0}{d} = 0.5\mu F$$

$$\epsilon = 3\epsilon_0$$

$$C_{eq} = \frac{A\epsilon}{2d} + \frac{A\epsilon_0}{2d}$$

$$= \frac{3A\epsilon_0}{2d} + \frac{A\epsilon_0}{2d}$$

$$= \frac{4A\epsilon_0}{2d}$$

$$= 2\left(\frac{A\epsilon_0}{d}\right)$$

$$= \boxed{1\mu F}$$